Rational numbers with purely periodic beta-expansion

Boris Adamczeswki, C. Frougny, A. Siegel, W.Steiner

Fractals, Tilings, and Things?

Fractals, Tilings, and Things ?

Number theory : expansions in several bases

Fractals,

Tilings,

and Things ?

Rauzy fractals

Number theory : expansions in several bases

Fractals,

Rauzy fractals

Tilings,

Self-affine tilings

and Things ?

Number theory : expansions in several bases

Fractals,

Rauzy fractals

Tilings,

Self-affine tilings

and Things ?

Number theory : expansions in several bases

Goal : **using fractal geometry to prove irrationality**

(Easy) questions from number theory

Who has a regular expansion in base 10 ?

(Easy) questions from number theory

Who has a regular expansion in base 10 ?

Greedy algorithm :

$$x = \frac{1}{10}(\underbrace{[10x]}_{=a_1} + \underbrace{\{10x\}}_{=x_1})$$

Who has a regular expansion in base 10 ?

Greedy algorithm :

$$
\begin{aligned}
x &= \tfrac{1}{10}(\underbrace{[10x]}_{=a_1} + \underbrace{\{10x\}}_{=x_1}) \\
&= \tfrac{a_1}{10} + \tfrac{1}{100}(\underbrace{[10x_1]}_{=a_2} + \underbrace{\{10x_1\}}_{=x_2}) \\
&= \tfrac{a_1}{10} + \tfrac{a_2}{10^2} + \cdots + \tfrac{a_n}{10^n} + \ldots
\end{aligned}
$$

(Easy) questions from number theory

Who has a regular expansion in base 10 ?

Greedy algorithm :

$$\begin{aligned} x &= \frac{1}{10}(\underbrace{[10x]}_{=a_1} + \underbrace{\{10x\}}_{=x_1}) \\ &= \frac{a_1}{10} + \frac{1}{100}(\underbrace{[10x_1]}_{=a_2} + \underbrace{\{10x_1\}}_{=x_2}) \\ &= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \ldots \end{aligned}$$

- What are the expansions in $[0, 1[$?
- Who has an eventually periodic expansion ?
- Who has a finite expansion ?
- Who has a purely periodic expansion ?

(Easy) questions from number theory

Who has a regular expansion in base 10 ?

Greedy algorithm :

$$\begin{aligned} x &= \frac{1}{10}(\underbrace{[10x]}_{=a_1} + \underbrace{\{10x\}}_{=x_1}) \\ &= \frac{a_1}{10} + \frac{1}{100}(\underbrace{[10x_1]}_{=a_2} + \underbrace{\{10x_1\}}_{=x_2}) \\ &= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \dots \end{aligned}$$

- What are the expansions in $[0, 1[$?
  Sequences in $\{0, 1 \dots 9\}^{\mathbb{N}}$ except those ending with 99999.....
  $$\text{Since } 1 = \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \dots$$
- Who has an eventually periodic expansion ?
- Who has a finite expansion ?
- Who has a purely periodic expansion ?

(Easy) questions from number theory

Who has a regular expansion in base 10 ?

Greedy algorithm :

$$\begin{aligned} x &= \frac{1}{10}(\underbrace{[10x]}_{=a_1} + \underbrace{\{10x\}}_{=x_1}) \\ &= \frac{a_1}{10} + \frac{1}{100}(\underbrace{[10x_1]}_{=a_2} + \underbrace{\{10x_1\}}_{=x_2}) \\ &= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \dots \end{aligned}$$

- What are the expansions in $[0, 1[$?
  Sequences in $\{0, 1 \dots 9\}^{\mathbb{N}}$ except those ending with 99999.....
  $$\text{Since } 1 = \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \dots$$
- Who has an eventually periodic expansion ?
  All rational numbers.
- Who has a finite expansion ?
- Who has a purely periodic expansion ?

(Easy) questions from number theory

Who has a regular expansion in base 10 ?

Greedy algorithm :

$$
\begin{aligned}
x &= \tfrac{1}{10}(\underbrace{[10x]}_{=a_1} + \underbrace{\{10x\}}_{=x_1}) \\
&= \tfrac{a_1}{10} + \tfrac{1}{100}(\underbrace{[10x_1]}_{=a_2} + \underbrace{\{10x_1\}}_{=x_2}) \\
&= \tfrac{a_1}{10} + \tfrac{a_2}{10^2} + \cdots + \tfrac{a_n}{10^n} + \ldots
\end{aligned}
$$

- What are the expansions in $[0,1[$?
  Sequences in $\{0,1\ldots 9\}^{\mathbb{N}}$ except those ending with 99999.....
  Since $1 = \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \ldots$
- Who has an eventually periodic expansion ?
  All rational numbers.
- Who has a finite expansion ?
  Rational numbers with denominator 10^n.
- Who has a purely periodic expansion ?

(Easy) questions from number theory

Who has a regular expansion in base 10 ?

Greedy algorithm :

$$\begin{aligned} x &= \frac{1}{10}(\underbrace{[10x]}_{=a_1} + \underbrace{\{10x\}}_{=x_1}) \\ &= \frac{a_1}{10} + \frac{1}{100}(\underbrace{[10x_1]}_{=a_2} + \underbrace{\{10x_1\}}_{=x_2}) \\ &= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \ldots \end{aligned}$$

- What are the expansions in $[0,1[$?
  Sequences in $\{0,1\ldots9\}^{\mathbb{N}}$ except those ending with 99999.....
  $$\text{Since } 1 = \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \ldots$$
- Who has an eventually periodic expansion ?
  All rational numbers.
- Who has a finite expansion ?
  Rational numbers with denominator 10^n.
- Who has a purely periodic expansion ?
  Rational number whose denominator is coprime with 10.

(Easy) questions from number theory

Who has a regular expansion in base 10 ?

Greedy algorithm :

$$\begin{aligned} x &= \frac{1}{10}(\underbrace{[10x]}_{=a_1} + \underbrace{\{10x\}}_{=x_1}) \\ &= \frac{a_1}{10} + \frac{1}{100}(\underbrace{[10x_1]}_{=a_2} + \underbrace{\{10x_1\}}_{=x_2}) \\ &= \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \ldots \end{aligned}$$

- What are the expansions in $[0,1[$?
  Sequences in $\{0,1\ldots 9\}^{\mathbb{N}}$ except those ending with 99999.....
  $$\text{Since } 1 = \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} + \ldots$$
- Who has an eventually periodic expansion ?
  All rational numbers.
- Who has a finite expansion ?
  Rational numbers with denominator 10^n.
- Who has a purely periodic expansion ?
  Rational number whose denominator is coprime with 10.

Remark : Numbers with a regular expansion are **uniformly spread in $[0,1]$**
(similar results hold for any integer base b)

Regular expansions in the golden ratio basis ?

Golden ratio : $\phi = \frac{1+\sqrt{5}}{2}$

Greedy algorithm : ϕ-expansions

$$x = \frac{1}{\phi}(\underbrace{[\phi x]}_{=a_1} + \underbrace{\{\phi x\}}_{=x_1})$$

Regular expansions in the golden ratio basis ?

Golden ratio : $\phi = \frac{1+\sqrt{5}}{2}$

Greedy algorithm : ϕ-expansions

$$
\begin{aligned}
x &= \frac{1}{\phi}(\underbrace{[\phi x]}_{=a_1} + \underbrace{\{\phi x\}}_{=x_1}) \\
&= \frac{a_1}{\phi} + \frac{1}{\phi}(\underbrace{[\phi x_1]}_{=a_2} + \underbrace{\{\phi x_1\}}_{=x_2}) \\
&= \frac{a_1}{\phi} + \frac{a_2}{\phi^2} + \cdots + \frac{a_n}{\phi^n} + \ldots
\end{aligned}
$$

Regular expansions in the golden ratio basis ?

Golden ratio : $\phi = \frac{1+\sqrt{5}}{2}$

Greedy algorithm : ϕ-expansions

$$\begin{aligned} x &= \frac{1}{\phi}(\underbrace{[\phi x]}_{=a_1} + \underbrace{\{\phi x\}}_{=x_1}) \\ &= \frac{a_1}{\phi} + \frac{1}{\phi}(\underbrace{[\phi x_1]}_{=a_2} + \underbrace{\{\phi x_1\}}_{=x_2}) \\ &= \frac{a_1}{\phi} + \frac{a_2}{\phi^2} + \cdots + \frac{a_n}{\phi^n} + \ldots \end{aligned}$$

- What are the expansions in $[0,1[$?
- Who has an eventually periodic expansion ?
- Who has a finite expansion ?
- Which **rational numbers** have a purely periodic expansion ?

Regular expansions in the golden ratio basis ?

Golden ratio : $\phi = \frac{1+\sqrt{5}}{2}$

Greedy algorithm : ϕ-expansions

$$\begin{aligned} x &= \frac{1}{\phi}(\underbrace{[\phi x]}_{=a_1} + \underbrace{\{\phi x\}}_{=x_1}) \\ &= \frac{a_1}{\phi} + \frac{1}{\phi}(\underbrace{[\phi x_1]}_{=a_2} + \underbrace{\{\phi x_1\}}_{=x_2}) \\ &= \frac{a_1}{\phi} + \frac{a_2}{\phi^2} + \cdots + \frac{a_n}{\phi^n} + \ldots \end{aligned}$$

- What are the expansions in $[0, 1[$?
  Sequences in $\{0, 1\}^{\mathbb{Z}}$ with no two consecutive 1 and not ending with 01010101....
  $$\left(\text{since } 1 = \frac{1}{\phi} + \frac{1}{\phi^2} = \frac{1}{\phi} + \frac{0}{\phi^2} + \frac{1}{\phi^3} + \frac{0}{\phi^3} + \ldots\right)$$
- Who has an eventually periodic expansion ?
- Who has a finite expansion ?
- Which **rational numbers** have a purely periodic expansion ?

Regular expansions in the golden ratio basis ?

Golden ratio : $\phi = \frac{1+\sqrt{5}}{2}$

Greedy algorithm : ϕ-expansions

$$\begin{aligned} x &= \frac{1}{\phi}(\underbrace{[\phi x]}_{=a_1} + \underbrace{\{\phi x\}}_{=x_1}) \\ &= \frac{a_1}{\phi} + \frac{1}{\phi}(\underbrace{[\phi x_1]}_{=a_2} + \underbrace{\{\phi x_1\}}_{=x_2}) \\ &= \frac{a_1}{\phi} + \frac{a_2}{\phi^2} + \cdots + \frac{a_n}{\phi^n} + \ldots \end{aligned}$$

- What are the expansions in $[0,1[$?
  Sequences in $\{0,1\}^{\mathbb{Z}}$ with no two consecutive 1 and not ending with 01010101....
  $$(\text{since } 1 = \frac{1}{\phi} + \frac{1}{\phi^2} = \frac{1}{\phi} + \frac{0}{\phi^2} + \frac{1}{\phi^3} + \frac{0}{\phi^3} + \ldots)$$
- Who has an eventually periodic expansion ?
  All $x \in \mathbb{Q}(\phi)$ *(Bertrand :77)*
- Who has a finite expansion ?
- Which **rational numbers** have a purely periodic expansion ?

Regular expansions in the golden ratio basis ?

Golden ratio : $\phi = \frac{1+\sqrt{5}}{2}$

Greedy algorithm : ϕ-expansions

$$\begin{aligned} x &= \frac{1}{\phi}(\underbrace{[\phi x]}_{=a_1} + \underbrace{\{\phi x\}}_{=x_1}) \\ &= \frac{a_1}{\phi} + \frac{1}{\phi}(\underbrace{[\phi x_1]}_{=a_2} + \underbrace{\{\phi x_1\}}_{=x_2}) \\ &= \frac{a_1}{\phi} + \frac{a_2}{\phi^2} + \cdots + \frac{a_n}{\phi^n} + \ldots \end{aligned}$$

- What are the expansions in $[0,1[$?
  Sequences in $\{0,1\}^{\mathbb{Z}}$ with no two consecutive 1 and not ending with 01010101....
  $$\left(\text{since } 1 = \frac{1}{\phi} + \frac{1}{\phi^2} = \frac{1}{\phi} + \frac{0}{\phi^2} + \frac{1}{\phi^3} + \frac{0}{\phi^3} + \ldots\right)$$
- Who has an eventually periodic expansion ?
  All $x \in \mathbb{Q}(\phi)$ *(Bertrand :77)*
- Who has a finite expansion ?

  All $x \in \mathbb{Z}[\phi]$
- Which **rational numbers** have a purely periodic expansion ?

Regular expansions in the golden ratio basis ?

Golden ratio : $\phi = \frac{1+\sqrt{5}}{2}$

Greedy algorithm : ϕ-expansions

$$\begin{aligned} x &= \frac{1}{\phi}(\underbrace{[\phi x]}_{=a_1} + \underbrace{\{\phi x\}}_{=x_1}) \\ &= \frac{a_1}{\phi} + \frac{1}{\phi}(\underbrace{[\phi x_1]}_{=a_2} + \underbrace{\{\phi x_1\}}_{=x_2}) \\ &= \frac{a_1}{\phi} + \frac{a_2}{\phi^2} + \cdots + \frac{a_n}{\phi^n} + \dots \end{aligned}$$

- What are the expansions in $[0,1[$?
  Sequences in $\{0,1\}^{\mathbb{Z}}$ with no two consecutive 1 and not ending with 01010101....
  $$\left(\text{since } 1 = \frac{1}{\phi} + \frac{1}{\phi^2} = \frac{1}{\phi} + \frac{0}{\phi^2} + \frac{1}{\phi^3} + \frac{0}{\phi^3} + \dots\right)$$
- Who has an eventually periodic expansion ?
  All $x \in \mathbb{Q}(\phi)$ *(Bertrand :77)*
- Who has a finite expansion ?

  All $x \in \mathbb{Z}[\phi]$
- Which **rational numbers** have a purely periodic expansion ?
  All $x \in \mathbb{Q} \cap [0,1]$ *(Schmidt :80)*

Regular expansions in the golden ratio basis ?

Golden ratio : $\phi = \frac{1+\sqrt{5}}{2}$

Greedy algorithm : ϕ-expansions

$$\begin{aligned} x &= \frac{1}{\phi}(\underbrace{[\phi x]}_{=a_1} + \underbrace{\{\phi x\}}_{=x_1}) \\ &= \frac{a_1}{\phi} + \frac{1}{\phi}(\underbrace{[\phi x_1]}_{=a_2} + \underbrace{\{\phi x_1\}}_{=x_2}) \\ &= \frac{a_1}{\phi} + \frac{a_2}{\phi^2} + \cdots + \frac{a_n}{\phi^n} + \dots \end{aligned}$$

- What are the expansions in $[0, 1[$?
  Sequences in $\{0, 1\}^{\mathbb{Z}}$ with no two consecutive 1 and not ending with 01010101....
  $$\left(\text{since } 1 = \frac{1}{\phi} + \frac{1}{\phi^2} = \frac{1}{\phi} + \frac{0}{\phi^2} + \frac{1}{\phi^3} + \frac{0}{\phi^3} + \dots\right)$$
- Who has an eventually periodic expansion ?
  All $x \in \mathbb{Q}(\phi)$ *(Bertrand :77)*
- Who has a finite expansion ?
  All $x \in \mathbb{Z}[\phi]$
- Which **rational numbers** have a purely periodic expansion ?
  All $x \in \mathbb{Q} \cap [0, 1]$ *(Schmidt :80)*

Still uniformly spread in $[0, 1]$

Regular expansions in the ~~golden ratio~~ **quadratic unit** basis ?

~~Golden ratio~~ **Base β : $\beta^2 = a\beta + b$** $b = \pm 1$ $\beta > 1$

Greedy algorithm : ~~ϕ-expansions~~ **β-expansions**

$$\begin{aligned} x &= \frac{1}{\beta}(\underbrace{[\beta x]}_{=a_1} + \underbrace{\{\beta x\}}_{=x_1}) \\ &= \frac{a_1}{\beta} + \frac{1}{\beta}(\underbrace{[\beta x_1]}_{=a_2} + \underbrace{\{\beta x_1\}}_{=x_2}) \\ &= \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \dots \end{aligned}$$

- What are the expansions in $[0,1[$?
  Sequences in $\{0,1\}^{\mathbb{N}}$ with no two consecutive 1 and not ending with 01010101.... **? ? ?**
  $$\left(\text{since } 1 = \cancel{\frac{1}{\phi} + \frac{1}{\phi^2}} = \frac{1}{\phi} + \frac{0}{\phi^2} + \frac{1}{\phi^3} + \frac{0}{\phi^3} + \dots\right)$$
- Who has an eventually periodic expansion ?
  All $x \in \mathbb{Q}(\phi)$ *(Bertrand :77)* **? ? ?**
- Who has a finite expansion ?
  All $x \in \mathbb{Z}[\phi]$ **? ? ?**
- Which **rational numbers** have a purely periodic expansion ?
  All $x \in \mathbb{Q} \cap [0,1]$ *(Schmidt :80)* **? ? ?**

Still uniformly spread in $[0,1]$ **? ?**

Regular expansions in the ~~golden ratio~~ **quadratic unit** basis ?

~~Golden ratio~~ **Base β : $\beta^2 = a\beta + b$** $b = \pm 1$ $\beta > 1$

Greedy algorithm : ~~ϕ-expansions~~ **β-expansions**

$$\begin{aligned} x &= \frac{1}{\beta}(\underbrace{[\beta x]}_{=a_1} + \underbrace{\{\beta x\}}_{=x_1}) \\ &= \frac{a_1}{\beta} + \frac{1}{\beta}(\underbrace{[\beta x_1]}_{=a_2} + \underbrace{\{\beta x_1\}}_{=x_2}) \\ &= \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \ldots \end{aligned}$$

- What are the expansions in $[0, 1[$?
  Sequences in $\{0, 1\}^{\mathbb{N}}$ ~~with no two consecutive 1~~ ~~and not ending with 01010101....~~ **Strictly smaller than the infinite expansion of 1.**
  (since $1 = \frac{a_1}{\beta} + \frac{a_2}{\beta} + \frac{a_3}{\beta^3} + \ldots$)
- Who has an eventually periodic expansion ?
  All $x \in \mathbb{Q}(\phi)$ *(Bertrand :77)* **Always true**
- Who has a finite expansion ?
  All $x \in \mathbb{Z}[\phi]$ **def := β satisfies the property (F) ; depends on $b = \pm 1$**
- Which **rational numbers** have a purely periodic expansion ?
  All $x \in \mathbb{Q} \cap [0, 1]$ *(Schmidt :80)* **all or none depending on (F)**

Still uniformly spread in $[0, 1]$ **(OK) + Dichotomy according to (F)**

Regular expansions in **any unit Pisot** basis ?

Base β : $\beta^n = a_{n-1}\beta^{n-1} + \cdots + \pm 1 \qquad \beta > 1$

Pisot hypothesis : $\beta > 1$ and its Galois conjugates $\beta^{(i)}$ are all strictly smaller than one.

Greedy algorithm : β-**expansions**

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \ldots$$

- What are the expansions in $[0, 1[$?
  Sequences in $\{0, 1\}^{\mathbb{N}}$ strictly smaller than the infinite expansion of 1. **? ?**
- Who has an eventually periodic expansion ?
  All $x \in \mathbb{Q}(\phi)$ *(Bertrand :77)* **? ?**
- Who has a finite expansion ?
  All $x \in \mathbb{Z}[\phi]$ (def := β satisfies the property (F)) **? ?**
- Which rational numbers have a purely periodic expansion ?
  All $x \in \mathbb{Q} \cap [0, 1]$ **? ?**

Still uniformly spread in $[0, 1]$ **? ?**

Regular expansions in **any unit Pisot** basis ?

Base β **:** $\beta^n = a_{n-1}\beta^{n-1} + \cdots + \pm 1 \qquad \beta > 1$

Pisot hypothesis : $\beta > 1$ and its Galois conjugates $\beta^{(i)}$ are all strictly smaller than one.

Greedy algorithm : β-**expansions**

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \dots$$

- What are the expansions in $[0, 1[$?
  Sequences in $\{0,1\}^{\mathbb{N}}$ strictly smaller than the infinite expansion of 1. **? ?**
  OK from Pisot assumption
- Who has an eventually periodic expansion ?
  All $x \in \mathbb{Q}(\phi)$ *(Bertrand :77)* **? ?** OK from Pisot assumption
- Who has a finite expansion ?
  All $x \in \mathbb{Z}[\phi]$ (def := β satisfies the property (F)) **? ?**
  Conditions for (F) *(Frougny&Solomyak :02, Akiyama :02...)*
- Which rational numbers have a purely periodic expansion ?
  All $x \in \mathbb{Q} \cap [0, 1]$ **? ?** **NO !**

Still uniformly spread in $[0, 1]$ **? ? NO !**

Regular expansions in **any unit Pisot** basis ?

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \ldots$$

Which rational numbers have a purely periodic expansion ?

Theorem (Akiyama :02)

If $\beta^3 = \beta + 1$, *there exists* $\gamma(\beta) = 0.666\,666\,666\,086\ \ldots$ *such that*

- *all* $x \in \mathbb{Q} \cap [0, \gamma(\beta)[$ *have a purely periodic* β*-expansion*
- *a sequence* $x_n \in \mathbb{Q}$, $\lim x_n = \gamma(\beta)$ *does not have a periodic* β*-expansion.*

Purely periodic expansions are NOT uniformly spread in $[0, 1]$

Regular expansions in **any unit Pisot** basis ?

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \ldots$$

Which rational numbers have a purely periodic expansion ?

Theorem (Akiyama :02)

If $\beta^3 = \beta + 1$, *there exists* $\gamma(\beta) = 0.666\,666\,666\,086\,\ldots$ *such that*

- *all* $x \in \mathbb{Q} \cap [0, \gamma(\beta)[$ *have a purely periodic* β*-expansion*
- *a sequence* $x_n \in \mathbb{Q}$, $\lim x_n = \gamma(\beta)$ *does not have a periodic* β*-expansion.*

Purely periodic expansions are NOT uniformly spread in $[0, 1]$

Three new questions.

- Is there still a dichotomy according to (F) ?
- What can be said on $\gamma(\beta)$?
- Is it the good conference : *Tilings* and *fractals* ? ? ? ? ?

Fractal in beta-numeration ?

Expand any positive real in base β

$$x = \underbrace{a_{-k+1}\beta^{k-1} + \cdots + a_{-1}\beta + a_0}_{\beta\text{-integral part}} + \underbrace{a_1\beta^{-1} + a_2\beta^{-2} + \cdots}_{\beta\text{-fractional part}}.$$

Compact representation of number with no fractional part ?

Fractal in beta-numeration ?

Expand any positive real in base β

$$x = \underbrace{a_{-k+1}\beta^{k-1} + \cdots + a_{-1}\beta + a_0}_{\beta\text{-integral part}} + \underbrace{a_1\beta^{-1} + a_2\beta^{-2} + \cdots}_{\beta\text{-fractional part}}.$$

Compact representation of number with no fractional part ?

- Galois conjugates of β $r-1$ real conjugates and $2s$ complex conjugates.
  Pisot assumption : $|\beta^{(i)}| < 1$

Example : smallest Pisot number $\beta^3 = \beta + 1$. Two complex conjugates $\beta^{(2)}, \overline{\beta^{(2)}}$.

Fractal in beta-numeration ?

Expand any positive real in base β

$$x = \underbrace{a_{-k+1}\beta^{k-1} + \cdots + a_{-1}\beta + a_0}_{\beta\text{-integral part}} + \underbrace{a_1\beta^{-1} + a_2\beta^{-2} + \cdots}_{\beta\text{-fractional part}}.$$

Compact representation of number with no fractional part ?

- Galois conjugates of β $r-1$ real conjugates and $2s$ complex conjugates.

  Pisot assumption : $|\beta^{(i)}| < 1$

Example : smallest Pisot number $\beta^3 = \beta + 1$. Two complex conjugates $\beta^{(2)}, \overline{\beta^{(2)}}$.

- Canonical embedding : **replace every polynomial in β by its conjugates**

$$\begin{array}{rcl} \xi : \mathbb{Q}(\beta) & \rightarrow & \mathbb{R}^{r-1} \times \mathbb{C}^s \\ x & \mapsto & (\sigma_2(x), \ldots, \sigma_{r+s}(x)). \end{array}$$

Smallest Pisot number : $\xi : P(\beta) \in \mathbb{R} \mapsto P(\beta^{(2)}) \in \mathbb{C}$

Fractal in beta-numeration ?

Expand any positive real in base β

$$x = \underbrace{a_{-k+1}\beta^{k-1} + \cdots + a_{-1}\beta + a_0}_{\beta\text{-integral part}} + \underbrace{a_1\beta^{-1} + a_2\beta^{-2} + \cdots}_{\beta\text{-fractional part}}.$$

Compact representation of number with no fractional part ?

- Galois conjugates of β $r-1$ real conjugates and $2s$ complex conjugates.

  Pisot assumption : $|\beta^{(i)}| < 1$

Example : smallest Pisot number $\beta^3 = \beta + 1$. Two complex conjugates $\beta^{(2)}, \overline{\beta^{(2)}}$.

- Canonical embedding : **replace every polynomial in β by its conjugates**

$$\begin{array}{rccc} \xi : & \mathbb{Q}(\beta) & \rightarrow & \mathbb{R}^{r-1} \times \mathbb{C}^s \\ & x & \mapsto & (\sigma_2(x), \ldots, \sigma_{r+s}(x)). \end{array}$$

Smallest Pisot number : $\quad \xi : P(\beta) \in \mathbb{R} \mapsto P(\beta^{(2)}) \in \mathbb{C}$

- **Central tile** (*Rauzy :81, Thurston :89, Akiyama :98*)

$$\mathcal{T} := \overline{\{\xi(\text{integral part of a positive number})\}}$$

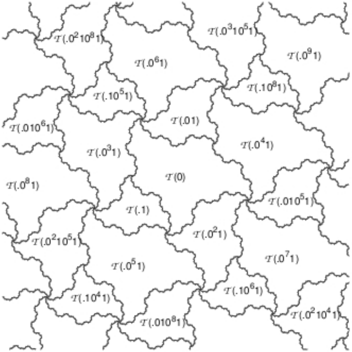
Smallest PP : $\quad \xi(\beta^n) = (\beta^{(2)})^n \in \mathcal{T}$, $\xi(\beta^5 + 1) = \beta^{(5)} + 1 \in \mathcal{T}$

but $\beta^2 + 1$ is not an integer part : $(\beta^{(2^2} + 1 \notin \mathcal{T}$?

Replace polynomials in β by polynomials in its conjugates and take the closure of reals with no fractional part

Tilings !

$$x \in \mathbb{Z}[\beta] \quad \mathcal{T}(x) := \overline{\{\xi(\text{reals with the same fractional part as } x)\}}$$



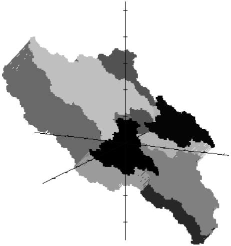
- Finite number of tiles (admissibility condition)
- Covering of the space

Theorem (Akiyama&Rao&Steiner :06)

Let β be a unit cubic Pisot number. Then the covering by central tiles is a tiling.

Fractals and things ?

Strong relation between periodicity and the central tile



- Property (F) iff 0 is an inner point *(Akiyama :02)*.

Fractals and things ?

Strong relation between periodicity and the central tile

- Property (F) iff 0 is an inner point *(Akiyama :02)*.
- Purely periodic expansions : searching for a natural extension
  - subdivise the central tile according to the admissibility condition
  - build a suspension

Fractals and things ?

Strong relation between periodicity and the central tile

- Property (F) iff 0 is an inner point *(Akiyama :02)*.
- Purely periodic expansions : searching for a natural extension
  - subdivise the central tile according to the admissibility condition
  - build a suspension

Ito&Rao :04 $x \in \mathbb{Q}(\beta)$ **purely periodic roughly means that** $-\xi(x)$ **lies in the central tile.**

Theorem (Ito&Rao :04)

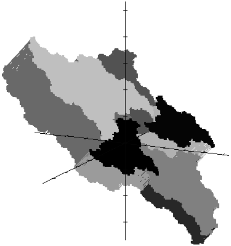
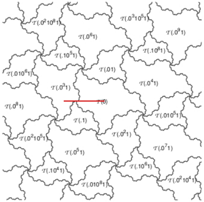
Let β be a Pisot unit and $x \in [0,1)$. The β-expansion of x is purely periodic if and only if $x \in \mathbb{Q}(\beta)$ and

$$(-\xi(x), x) \in \mathcal{E}_\beta := \bigcup_{i=0}^{n+m-1} \mathcal{T}_i \times [0, T_\beta^i(1)).$$

Back to purely periodic expansions

Reminder 1 $x \in \mathbb{Q}(\beta)$ purely periodic means that $-\xi(x)$ lies in the central tile.

Reminder 2 $\gamma(\beta)$ is the lenght of the largest interval issued from 0 that contains only purely periodic expansions

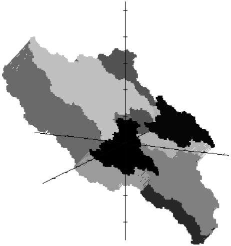



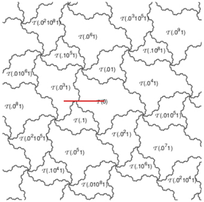
New question : **What is the largest diagonal contained in the suspension ?**

Back to purely periodic expansions

Reminder 1 $x \in \mathbb{Q}(\beta)$ purely periodic means that $-\xi(x)$ lies in the central tile.

Reminder 2 $\gamma(\beta)$ is the lenght of the largest interval issued from 0 that contains only purely periodic expansions





New question : **What is the largest diagonal contained in the suspension ?**

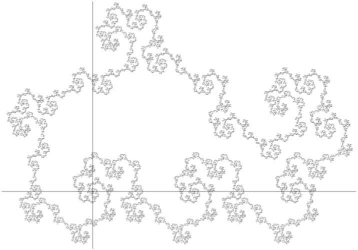
Corollary

When the tiling property holds, $\gamma(\beta)$ lies at the intersection of two tiles and a rational line in the tiling.

Self-similar properties

Definition (Spiral point)

A point $z \in \mathcal{T}$ is a *spiral point* if for all ε and θ, both $Int(\mathcal{T})$ and the complement $\mathcal{C}(\mathcal{T})$ meet the ray $z + [0, \varepsilon)e^{i\theta} := \left\{ z + \rho e^{i\theta} \mid \rho \in [0, \varepsilon) \right\}$.



Admissibility condition yields self-similar IFS and then spiral properties

Lemma

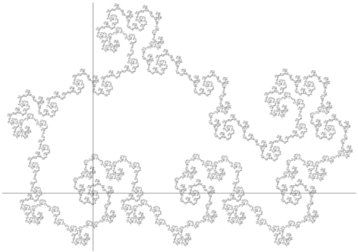
Let β be a cubic Pisot number with a complex Galois conjugate α.
Then every point in $\mathbb{Q}(\alpha)$ that belongs to the boundary of $\mathcal{T}$ is a spiral point with respect to this tile.

Dichotomy with respect to (F)

If 0 is on the boundary :

- The boundary has a spiral shape on 0
- Points on the horizontal line out of the central tile.

  Small rationals with non purely periodic expansion

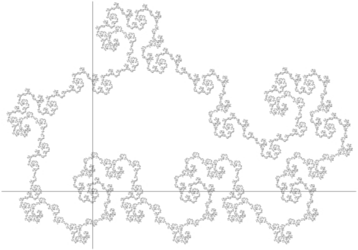


Dichotomy with respect to (F)

If 0 is on the boundary :

- The boundary has a spiral shape on 0
- Points on the horizontal line out of the central tile.

Small rationals with non purely periodic expansion



There is still a dichotomy in the cubic case

Theorem

Let β be a cubic Pisot unit. Then, one of the following holds :

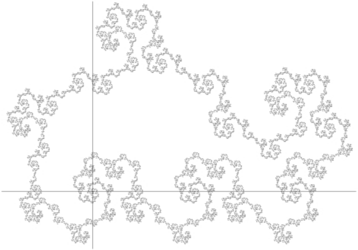
(i) β *satisfies* (F) *and* $\gamma(\beta) > 0$,

(ii) β *does not satisfy* (F) *and* $\gamma(\beta) = 0$.

Irrationality

If $\gamma(\beta)$ is rational :

- The boundary has a spiral shape on $-\gamma(\beta)$
- Points smaller than $\gamma(\beta)$ out of the central tile.
  The interval $-[0, \gamma(\beta)[$ *cannot be fully included in the tile*

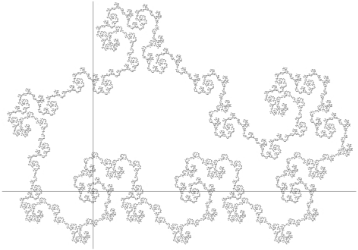


Irrationality

If $\gamma(\beta)$ is rational :

- The boundary has a spiral shape on $-\gamma(\beta)$
- Points smaller than $\gamma(\beta)$ out of the central tile.

  The interval $-[0,\gamma(\beta)[$ *cannot be fully included in the tile*



The "086" in $\gamma(\beta)$ was not a computational error
Purely periodic expansion are definitively not randomly spread in $[0,1[$

Theorem

Let β be a cubic Pisot unit satisfying (F) and such that the number field $\mathbb{Q}(\beta)$ is not totally real. Then, $\gamma(\beta)$ is irrational. In particular, $0<\gamma(\beta)<1$.

The beginning of a long history ?

Fractal and tilings bring unexpected proofs of irrationality

- Non cubic case self-affine structure instead of self-similar.

  Intersection of a line with a the tile ? Shape of the boundary ?

- Non unit case. Suspension with p-adic representations.

  No more dichotomy with respect to (F).

- Fractals and numeration
  - Best simultaneous approximations : *largest ball in the fractal.*
  - Radix expansions : hierarchy of IFS. *Which property remains ?*
  - Tiling condition ?