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A globally adaptive explicit numerical method for exploding systems of ordinary differential equations ☆

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ABSTRACT

This paper considers the mathematical framework of a sliced-time computation method for explosive solutions to systems of ordinary differential equations: $Y(t) \in \mathbb{R}^k$: $\frac{dY}{dt} = F(Y)$, $0 < t$, $Y(0) = Y_0$, that have **finite or infinite explosion time**. The method used generates automatically a sequence of non-uniform slices $\{[T_{n-1}, T_n] \mid n \geqslant 1\}$ determined by an end-of-slice condition that controls the growth of the solution within each slice. It also uses rescaling of the variables, whereas: $t = T_{n-1} + \beta_n s$ and $Y(t) = Y(T_{n-1}) + D_n Z_n(s)$, $D_n \in \mathbb{R}^{k\times k}$, and β_n being respectively an invertible diagonal matrix and a rescaling time factor. Thus, the original system is transformed into a sequence of slices-dependent initial-value shooting problems: $\frac{dZ_n}{ds} = G_n(Z_n)$, $0 < s \leqslant s_n$, $Z_n(0) = 0$, $\|Z_n(s)\| \leqslant S$ and $\|Z_n(s_n)\| = S$, where S is a threshold value and $\|.\|$ is the infinity norm on $\mathbb{R}^k$. A suitable selection of β_n and D_n leads the rescaled systems to satisfy a concept of **uniform similarity**, allowing to disable the extreme stiffness of the original ODE problem. Then, on each time slice, the uniformly rescaled systems are locally solved using a 4th order explicit Runge–Kutta scheme, within a computational tolerance of ϵ_{loc}. The sequential implementation of the local solver on a total of N slices leads to approximating the solution $Y(t)$ of the original system within a global tolerance ϵ_{glob}.

The proper definition of uniform similarity leads to deriving, under a stability assumption, a relationship between ϵ_{loc}, ϵ_{glob} and N. Such relation does not appear to be a sharp one particularly for the case when the existence time is infinite. In fact, numerical experiments conducted for infinite and finite times explosive discrete reaction diffusion problems attest for better estimates and for efficiency of the method in terms of stability and accuracy.

1. Position of the problem. Time slicing and rescaling

In solving time-dependent initial value systems, challenging computational problems occur when the solution has an explosive behavior. In the past two decades, several authors have dealt with such issue. The idea of rescaling time-dependent partial differential equations problem has appeared in [1,7] and [5]. In [2] and [3], perturbation of the time-dependent partial differential equation in view of solving a non-explosive solution was successfully pursued.

In this paper the idea of rescaling is being considered on the basis of a method introduced in [8] to solve semi-discrete diffusion reaction partial differential equations. Such method has proved to be extremely efficient and was successfully

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used to solve a membrane second order ordinary differential equation in [9]. Our intention is to provide a mathematical framework that justifies the efficiency of the method and conduct experiments that justify the obtained theoretical results.

To start, we consider the problem of numerically integrating systems of ordinary differential equations, where one seeks $Y:[0,T)\to\mathbb{R}^k$, such that:

$$\frac{dY}{dt}=F(Y),\quad 0<t\leqslant T\leqslant\infty,\qquad Y_i(0)=Y_{0,i}>0,\quad \forall i, \tag{1}$$

where the function $F:\mathbb{R}^k\to\mathbb{R}^k\in C^1$ is *locally Lipschitz Continuous* and the solution $Y(t)$ has strictly positive components, a monotonously increasing l^∞ norm and an explosive behavior, i.e.:

$$\forall t\geqslant 0,\ \forall i,\quad Y_i(t)\geqslant\gamma_0>0,\qquad \big\|Y(t)\big\|\ \text{increasing and}\ \lim_{t\to T\leqslant\infty}\big\|Y(t)\big\|=\infty,\quad \text{where } \|.\|=\|.\|_\infty. \tag{2}$$

Remark 1. This assumption makes the results of this paper restricted to non-oscillatory blow-up problems although the proposed numerical method has been extended to oscillatory blow-up problems as shown in [9].

As for the location of the blow-up, we assume that $\exists i_0\in\{1,2,\ldots,k\}$ such that for t sufficiently large:

$$Y_{i_0}(t)=\big\|Y(t)\big\|; \tag{3}$$

i_0 could be unique (blow-up in one point) or could occur for all i (blow-up every where).

Furthermore, the existence interval $[0,T]$:

$$\text{may be finite, } T<\infty\,(\text{local existence})\quad\text{or}\quad\text{infinite } T=\infty\,(\text{global existence}). \tag{4}$$

Globally adaptive numerical integration is obtained by breaking (1) into a sequence of **initial value shooting problems**. Such **time slices computations** result from the introduction of an end-of-slice function that controls the growth of the solution $Y(t)$.

Specifically, let S be a positive number. We then introduce the first slice as follows:

$$\begin{cases}\dfrac{dY}{dt}=F(Y),\quad 0<t<T_1 & (5.1)\\ Y(0)=Y_0 & (5.2)\\ \dfrac{\|Y(T_1)-Y_0\|}{\|Y_0\|}=S\quad\text{and}\quad \dfrac{\|Y(t)-Y_0\|}{\|Y_0\|}<S,\quad 0<t<T_1 & (5.3)\end{cases} \tag{5}$$

Subsequently $Y_1=Y(T_1)$ becomes the initial condition for a 2nd slice... and more generally for $n>1$, Y_{n-1} the initial condition for the nth slice. Thus, a recurrence definition can be established to obtain any slice n.

Given $Y(t)$ on the first $n-1$ slices (and henceforth $\{Y_1=Y(T_1),\ldots,Y_{n-1}=Y(T_{n-1})\}$), the nth slice is defined through the Initial Value Shooting Problem (IVSP):

$$\begin{cases}\dfrac{dY}{dt}=F(Y),\quad T_{n-1}<t<T_n & (6.1)\\ Y(T_{n-1})=Y_{n-1} & (6.2)\\ E_n\big(Y(T_n)\big)=0\quad\text{and}\quad E_n\big(Y(t)\big)<0,\quad T_{n-1}<t<T_n & (6.3)\end{cases} \tag{6}$$

where:

$$E_n\big(Y(t)\big)=\frac{\|Y(t)-Y_{n-1}\|}{\|Y_{n-1}\|}-S,$$

is the end-of-slice function (6.3) is referred to as the end-of-slice condition (EOS). Such approach allows to break (5) into a sequence of Initial Values Shooting Problems, whereas the value Y_{n-1} of the solution at end-of-slice $n-1$, becomes the initial value of the solution at beginning of slice n, $\forall n=1,2,\ldots,N$.

This clearly leads to a **coarse grid generation**: $\{(0=T_0<T_1<\cdots<T_n<\cdots)\}$ with a corresponding sequence of vectors $\{Y_n=Y(T_n)\mid n=0,1,\ldots\}$.

On the basis of assumption (2), the sequence $\{\|Y_n\|\}$ satisfies the following property:

For all $S>0$:

$$\begin{cases}\|Y_0\|<\|Y_1\|<\cdots<\|Y_n\|<\cdots & (7.1)\\ \lim_{n\to\infty}\|Y_n\|=\infty & (7.2)\\ Y_{n,i}>0,\quad \forall i,n\,(\text{non-zero condition}) & (7.3)\end{cases} \tag{7}$$

An alternative definition for the end-of-slice function can be defined, specifically:

$$E_n^1\big(Y(t)\big) = \max_i \left\{ \frac{|Y_i(t) - Y_{n-1,i}|}{|Y_{n-1,i}|} \right\} - S = \max_i \left| \frac{Y_i(t)}{Y_{n-1,i}} - 1 \right| - S, \tag{8}$$

so that another coarse grid generation may be obtained through the sequence of problems (9):

$$\begin{cases} \dfrac{dY}{dt} = F(Y), \quad T_{n-1} < t < T_n & (9.1) \\ Y(T_{n-1}) = Y_{n-1} & (9.2) \\ E_n^1(Y_n) = 0 \quad \text{and} \quad E_n^1\big(Y(t)\big) < 0, \quad T_{n-1} < t < T_n & (9.3) \end{cases} \tag{9}$$

Given the explosive behavior of the solution, a first element of uniformity is introduced by rescaling each of the initial value shooting problems (9) on the basis of a change of variables whereby *the rescaled time variable s and rescaled solution $Z_n(s)$ are set to zero at the onset of every slice*. Specifically, for $t \in [T_{n-1}, T_n]$ and for every n, let:

$$\begin{cases} t = T_{n-1} + \beta_n s, & (10.1) \\ Y(t) = Y_{n-1} + D_n Z_n(s) = D_n\big(\mathbf{e} + Z_n(s)\big) & (10.2) \end{cases} \tag{10}$$

where $\beta_n > 0$ is the slice time scale; $\mathbf{e} \in \mathbb{R}^k$ is a vector which all components are equal to 1 and D_n is the diagonal matrix associated with vector Y_{n-1}:

$$D_{n,i,i} = Y_{n-1,i} > 0; \qquad D_{n,i,j} = 0, \quad i \neq j.$$

In scalar form (10.2) is equivalent to:

$$Y_i(t) = Y_{n-1,i}\big(1 + Z_{n,i}(s)\big), \quad i = 1, \ldots, k, \qquad T_{n-1} \leqslant t \leqslant T_n; \quad 0 \leqslant s \leqslant s_n.$$

Notation. For $W \in \mathbb{R}^k$, $W_i \neq 0$, we let $D(W)$ and $D^{-1}(W) \in \mathbb{R}^{k\times k}$ be the diagonal matrices which elements are:

$$\forall i, \quad \big(D(W)\big)_{ii} = W_i, \qquad \big(D^{-1}(W)\big)_{ii} = W_i^{-1}.$$

In that context $D_n = D(Y_{n-1})$.

Remark 2. In case $Y(t)$ may have some zero-components, the change of variables can still be $Y(t) = Y_{n-1} + D_n Z_n(s)$, as given in (10.2), but D_n is then defined as a diagonal invertible matrix, depending solely on the starting value Y_{n-1}, as follows:

$$D_{n,i,i} = \begin{cases} Y_{n-1,i} & \text{if } Y_{n-1,i} \neq 0, \\ 1 & \text{if } Y_{n-1,i} = 0. \end{cases} \tag{11}$$

This case has been tested on an application problem in [6].

The Initial Value Problem (1) is now equivalent to a sequence of rescaled Initial Value Shooting Problems in which one seeks on each nth slice, the pair: $\{s_n, Z_n : [0, s_n] \to \mathbb{R}^k\}$ such that:

$$\begin{cases} \dfrac{dZ_n}{ds} = G_n(Z_n), \quad 0 < s < s_n & (12.1) \\ Z_n(0) = 0 & (12.2) \\ H_n\big(Z_n(s_n)\big) = 0 \quad \text{and} \quad H_n(Z_n(s)) < 0, \quad 0 < s < s_n & (12.3) \end{cases} \tag{12}$$

where:

$$G_n(Z_n) = \beta_n D_n^{-1} F(Y_{n-1} + D_n Z_n)$$

and $H_n : \mathbb{R}^k \to \mathbb{R}$ is derived from either the function $E_n(.)$ (or $E_n^1(.,.)$) through the change of variables (10). Note that in the case when the end-of-slice condition function is (8), then the functions $\{H_n\}$ are uniformly independent from n, specifically:

$$H_n\big(Z_n(s)\big) = H\big(Z_n(s)\big) = \max_i \big\{ \big| Z_{n,i}(s) \big| \big\} - S = \big\| Z_n(s) \big\| - S.$$

1.1. Application problem

In this paper, our method is illustrated on the system of ODE's resulting from the semi-discretization of a reaction-diffusion Partial Differential Equation model. Specifically, let Ω be an open rectangular space. One seeks $\{u(x,t) \mid x \in \Omega,\ t \geqslant 0\}$ that satisfies:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m + a u^p, \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0,\ u(x,0) = u_0(x) > 0, \end{cases} \tag{13}$$

where $a > 0$, $p \geqslant 1$ and $0 < m \leqslant 1$ are constants.

Equivalently if $v = u^m$ and $q = \frac{1}{m}$, then the PDE in (13) is equivalent to:

$$\frac{\partial v}{\partial t} = \frac{1}{q} v^{1-q} \Delta v + \frac{a}{q} v^{pq-q+1}. \tag{14}$$

In particular when $m = 1$, (13) reduces to the classical semi-linear parabolic equation:

$$\frac{\partial v}{\partial t} = \Delta v + a v^p. \tag{15}$$

Note also that the linear case is obtained when $m = p = 1$.

Leroux and Maingé (1996) [4], have shown explosive positive behavior of the unique solution to (13) under restrictive conditions on u_0. Semi-discretization of (13) using a Finite Difference approach leads to the Initial Value Problem:

$$\frac{dY}{dt} = F(Y) = -D_1(Y) A Y + \phi(Y), \quad t > 0,\ Y(0) = Y_0, \tag{16}$$

where A is a positive definite matrix that discretizes $-\Delta$, $D_1(Y)$ and $\phi(Y)$ are respectively, a diagonal matrix discretizing $\frac{1}{q} v^{1-q}$ and a vector function discretizing $\frac{a}{q} v^{pq-q+1}$.

This Initial Value Problem has then been rescaled using the end-of-slice condition (8) and the change of variables (10).

The remaining part of this paper is divided as follows. In Section 2, we present the concept of uniform similarity between the rescaled Initial-Shooting Value problem systems (12). Based on additional assumptions on $Y(t)$, assumptions (17) and (18), we prove Theorem 1 that provides a justification for a selection of the time scale parameter β_n. We also show the validity of those assumptions in a case of the application problem (16). This is followed in Section 3 by introducing first the local explicit solver for (12) followed by the global solver for the solution $Y(t)$ of (1). Local and global estimates are then stated, particularly the global estimate of Theorem 3 based on a stability assumption (38). Also, an estimate on the total number of slices is given. In Section 4, proof of Theorem 3 is provided and the paper is completed by a section on numerical experiments conducted on a linear problem with infinite explosive time and a semi-linear explosive one with finite existence time. In the first one, knowledge of the exact solution allows to test the validity of the estimates being obtained, while tests conducted on the second one confirm the power of the method in reaching accurately blow-up times.

2. Uniform similarity. Additional assumptions and main results

Criteria for uniform similarity for (12) are given in what follows with $c < C$, **generic constants independent from** n.

Definition 1. The rescaled systems are said to be uniformly similar if the selection of $\{\beta_n\}$ is such that $\{Z_n(s), G_n(Z_n(s)), s_n\}$ in (12) satisfy:

1. $\forall n,\ \forall s \leqslant s_n\colon\ \|G_n(Z_n(s))\| \leqslant c$,
2. $\forall n,\ \forall s \leqslant s_n\colon\ \|G_{n,Z}(Z_n)\| \leqslant c$,
3. $\forall n\colon\ c_0 \leqslant s_n$,
4. $\forall n\colon\ s_n \leqslant c_1$,

where c_0 and c_1 are positive constants independent from n.

Additional assumptions on $Y(t)$ and $F(Y(t))$ are needed in order to have the rescaled systems satisfy these properties. Specifically, **there exists a constant** $\gamma \geqslant 1$, independent from n, such that:

As $n \to \infty$ and $\forall t \in [T_{n-1}, T_n]$, one has:

$$\begin{cases} c\|Y(t)\|^{\gamma} \leqslant \|F(Y(t))\| \leqslant C\|Y(t)\|^{\gamma}, \quad \text{i.e. } \|F(Y(t))\| \text{ is } \Theta(\|Y(t)\|^{\gamma}) & (17.1) \\ c\|Y(t)\|^{\gamma-1} \leqslant \|D_n^{-1} F(Y(t))\| \leqslant C\|Y(t)\|^{\gamma-1}, \quad \text{i.e. } \|D_n^{-1} F(Y(t))\| \text{ is } \Theta(\|Y(t)\|^{\gamma-1}) & (17.2) \\ \|D_n^{-1} F_Y(Y(t)) D_n\| \leqslant C\|Y(t)\|^{\gamma-1} \quad \text{i.e. } \|D_n^{-1} F_Y(Y(t)) D_n\| \text{ is } O(\|Y(t)\|^{\gamma-1}) & (17.3) \end{cases} \tag{17}$$

Also, at any blow-up location i_0 as given in (3), we assume that:

$$\text{As } n \to \infty,\ \forall t \in [T_{n-1}, T_n], \quad \big(Y_{n-1,i_0}(t)\big)^{-1}\big(F\big(Y(t)\big)\big)_{i_0} \geqslant c\big\|Y(t)\big\|^{\gamma-1}. \tag{18}$$

Theorem 1. *Let $S > 0$. If $\forall n \geqslant 1$ we select*:

$$\beta_n = \frac{1}{\|D_n^{-1}F(Y_{n-1})\|}, \tag{19}$$

then, under assumptions (17.2) *and* (17.3)*, the rescaled problems* (12) *satisfy properties* 1, 2 *and* 3 *of Definition* 1*. Furthermore, on the basis of* (18)*, property* 4 *of that definition is also satisfied.*

The proof of Theorem 1 is done through a sequence of lemmas.

Lemma 1. *For $\beta_n = \frac{1}{\|D_n^{-1}F(Y_{n-1})\|}$ and under assumption* (17.2) *on $F(.)$, one has*:

$$\forall n,\ \forall s \leqslant s_n: \quad \big\|G_n\big(Z_n(s)\big)\big\| \leqslant c(1+S)^{\gamma-1}.$$

Proof. Since $G_n(Z_n(s)) = \beta_n D_n^{-1}F(D_n(\mathbf{e}+Z_n(s))) = \frac{1}{\|D_n^{-1}F(Y_{n-1})\|}D_n^{-1}F(D_n(\mathbf{e}+Z_n(s))$, then one writes:

$$\big\|G_n\big(Z_n(s)\big)\big\| = \frac{1}{\|D_n^{-1}F(Y_{n-1})\|}\big\|D_n^{-1}F\big(D_n\big(\mathbf{e}+Z_n(s)\big)\big)\big\|.$$

Hence using (17.2) and knowing that $\|D_n\| = \|Y_{n-1}\|$ one has:

$$\big\|G_n\big(Z_n(s)\big)\big\| \leqslant c\left(\frac{\|D_n(\mathbf{e}+Z_n(s))\|}{\|Y_{n-1}\|}\right)^{\gamma-1} \leqslant c(1+S)^{\gamma-1}. \qquad \square$$

In a similar way we bound the Jacobian $G_{n,Z}(.)$. Specifically, we show now:

Lemma 2. *For $\beta_n = \frac{1}{\|D_n^{-1}F(Y_{n-1})\|}$ and under assumptions* (17.2) *and* (17.3)*, the $k \times k$ matrix $G_{n,Z}(Z_n(s))$ satisfies*:

$$\forall n,\ \forall s \leqslant s_n: \quad \big\|G_{n,Z}\big(Z_n(s)\big)\big\| \leqslant c(1+S)^{\gamma-1}.$$

Proof. Since $G_{n,Z}(Z_n(s)) = \beta_n D_n^{-1}F_Y(D_n(\mathbf{e}+Z_n(s)))D_n = \frac{1}{\|D_n^{-1}F(Y_{n-1})\|}D_n^{-1}F_Y(D_n(\mathbf{e}+Z_n(s)))D_n$, one has:

$$\big\|G_{n,Z}\big(Z_n(s)\big)\big\| = \frac{1}{\|D_n^{-1}F(Y_{n-1})\|}\big\|D_n^{-1}F_Y\big(D_n\big(\mathbf{e}+Z_n(s)\big)\big)D_n\big\|.$$

Hence using (17.2) and (17.3), one deduces:

$$\big\|G_{n,Z}\big(Z_n(s)\big)\big\| \leqslant c\left(\frac{\|D_n(\mathbf{e}+Z_n(s))\|}{\|Y_{n-1}\|}\right)^{\gamma-1} \leqslant c(1+S)^{\gamma-1}. \qquad \square$$

Thus Lemmas 1 and 2 lead to uniform similarity properties 1 and 2 given in Definition 1. We turn now to prove properties 3 and 4. The first is a direct consequence of Lemma 1. Specifically:

Lemma 3. *Under the assumptions of Lemma* 1*, the sequence $\{s_n\}$ is such that*:

$$c_0 = \frac{S}{c(1+S)^{\gamma-1}} \leqslant s_n.$$

Proof. This follows from:

$$Z_n(s) = \int_0^s G_n\big(Z_n\big(s'\big)\big)\,ds', \quad \forall s \leqslant s_n$$

which leads to:

$$\forall s, \quad \left\| Z_n(s) \right\| \leqslant \int_0^s \left\| G_n\left(Z_n\left(s'\right)\right) \right\| ds' \leqslant c(1+S)^{\gamma-1} s.$$

This yields at $s = s_n$:

$$S \leqslant c(1+S)^{\gamma-1} s_n \quad \Longleftrightarrow \quad \frac{S}{c(1+S)^{\gamma-1}} \leqslant s_n.$$

We finally handle Property 4 of the Uniform Similarity.

Lemma 4. *Under assumptions* (17.2) *and* (18)*, one has*:

$$s_n \leqslant \frac{S}{c} = c_1.$$

Proof. Consider the scalar function:

$$\left(G_n\left(Z_n(s)\right)\right)_{i_0} = \beta_n\left(Y_{n-1,i_0}^{-1}\right)\left(F\left(D_n\left(e + Z_n(s)\right)\right)_{i_0}.$$

From assumption (18), one has for large n:

$$\left(G_n\left(Z_n(s)\right)\right)_{i_0} \geqslant c\beta_n \left\| Y(t) \right\|^{\gamma-1} = c\beta_n \left(Y_{i_0}(t)\right)^{\gamma-1}.$$

Having also from assumption (17.2):

$$\beta_n = \frac{1}{\|D_n^{-1} F(Y_{n-1})\|} \geqslant \frac{c}{\|Y_{n-1}\|^{\gamma-1}} = \frac{c}{(Y_{n-1,i_0})^{\gamma-1}},$$

one deduces for large n and $t \in [T_{n-1}, T_n]$:

$$\left(G_n\left(Z_n(s)\right)\right)_{i_0} \geqslant \frac{c}{(Y_{n-1,i_0})^{\gamma-1}} \left(Y_{i_0}(t)\right)^{\gamma-1} = \frac{c}{(Y_{n-1,i_0})^{\gamma-1}} (Y_{n-1,i_0}\left(1 + Z_{n,i_0}(s)\right)^{\gamma-1} = c\left(1 + Z_{n,i_0}(s)\right)^{\gamma-1},$$

and since $Z_{n,i_0}(s) \geqslant 0$, because of the monotone increase of $\|Y(t)\| = Y_{i_0}(t)$, it follows:

$$\left(G_n\left(Z_n(s)\right)\right)_{i_0} \geqslant c.$$

Thus, by integration:

$$Z_{n,i_0}(s) = \int_0^s G_{n,i_0}\left(Z_n\left(s'\right)\right) ds' \geqslant cs, \quad \forall s \leqslant s_n.$$

At $s = s_n$ at which $Z_{n,i_0}(s) = S$, one has then: $S \geqslant cs_n \Longleftrightarrow s_n \leqslant \frac{S}{c}$.

Hence the result of Lemma 4.

The above lemmas provide and end the proof of Theorem 1. □

As a consequence of Theorem 1 and knowing that:

$$T_n = \sum_{i=1}^{n} \beta_i s_i,$$

one has the following result.

Corollary 1. *If the rescaled systems* (12) *are uniformly similar then the blow-up time is finite if and only if*:

$$\sum_{n=1}^{\infty} \beta_n < \infty.$$

Proof. This result follows from the inequality:

$$c_0 \sum_{i=1}^{n} \beta_i \leqslant T_n \leqslant c_1 \sum_{i=1}^{n} \beta_i$$

and the finite time blow-up occurring if and only if $\lim_{n\to\infty} T_n < \infty$. □

2.1. Validity of assumptions (17) and (18) in a case of the application problem (13)

We only consider in this paper the reaction–diffusion problem (13) with $m = 1$, $p \geqslant 1$, its semi-discretization using finite-difference approximations yielding the following system of ODE's:

$$\frac{dY}{dt} = F(Y) = -AY + \phi(Y), \tag{20}$$

where the matrix A discretizes Laplace's operator. Let:

$$\forall i \in \{1, 2, \ldots, k\}, \quad N(i) = \{j \mid j \neq i \text{ and } A_{ij} \neq 0\}$$

be the set of "neighboring" nodes to each node $i \in \{1, 2, \ldots, k\}$, then the matrix A satisfies the following properties:

- Sparsity: $card(N(i)) \leqslant l \ll k$,
- Diagonal dominance:

$$A_{ii} \geqslant \sum_{j \in N(i)} |A_{ij}| > 0, \quad \forall i \in \{1, 2, \ldots, k\}, \tag{21}$$

- with:

$$A_{ij} < 0 \quad \forall j \in N(i). \tag{22}$$

As for $\phi(Y(t))$, it is component-wise given by:

$$\forall i \in \{1, 2, \ldots, k\}, \quad \big(\phi\big(Y(t)\big)\big)_i = a\big(Y_i(t)\big)^p,$$

where the positive constant a is assumed to satisfy:

$$a > \max_i\{A_{ii}\} = a_0. \tag{23}$$

Let i_0 be the component(s) at which the blow-up is occurring, i.e. for n sufficiently large and $\forall t \in [T_{n-1}, T_n]$, $Y_{i_0}(t) = \|Y(t)\|$.

2.2. Validity of assumption (17.1)

From (20) one deduces:

$$\big\|F\big(Y(t)\big)\big\| = \big\|-AY(t) + \phi\big(Y(t)\big)\big\| \leqslant \|A\|\big\|Y(t)\big\| + a\big\|Y(t)\big\|^p,$$

with $p \geqslant 1$, yielding:

$$\frac{\|F(Y(t))\|}{\|Y(t)\|^p} \leqslant \frac{\|A\|}{\|Y(t)\|^{p-1}} + a \leqslant \frac{\|A\|}{\|Y_0\|^{p-1}} + a = constant. \tag{24}$$

On the other hand, since $(-AY(t))_i = -A_{ii}Y_i(t) - \sum_{j \in N(i)} A_{ij}Y_j(t)$, one gets:

$$\big(F\big(Y(t)\big)\big)_i = a\big(Y_i(t)\big)^p - A_{ii}Y_i(t) + \sum_{j \in N(i)} |A_{ij}|Y_j(t) \geqslant a\big(Y_i(t)\big)^p - A_{ii}Y_i(t).$$

Hence, using (23) one gets for all $i \in \{1, 2, \ldots, k\}$:

$$\big\|F\big(Y(t)\big)\big\| \geqslant \big(F\big(Y(t)\big)\big)_i \geqslant a\big(Y_i(t)\big)^p - a_0Y_i(t).$$

In particular, this inequality holds for $i = i_0$ where $Y_{i_0}(t) = \|Y(t)\|$ and yields:

$$\big\|F\big(Y(t)\big)\big\| \geqslant a\big\|Y(t)\big\|^p - a_0\big\|Y(t)\big\|.$$

Since $a_0 < a$, then there exists $\delta_1 > 0$ such that for n sufficiently large and $\forall t \in [T_{n-1}, T_n]$:

$$\frac{\|F(Y(t))\|}{\|Y(t)\|^p} \geqslant \delta_1 = constant. \tag{25}$$

Together, (24) and (25) validate assumption (17.1).

2.3. Validity of assumption (17.2)

Note that:

$$\left(D_n^{-1}F\left(Y(t)\right)\right)_i = a\frac{(Y_i(t))^p}{Y_{n-1,i}} - A_{ii}\frac{Y_i(t)}{Y_{n-1,i}} - \frac{1}{Y_{n-1,i}}\sum_{j\in N(i)} A_{ij}Y_j(t), \tag{26}$$

implying, since $Y_j(t) > 0$, $\forall j$ and $A_{ij} < 0$, $\forall j \in N(i)$:

$$\begin{aligned}\left(D_n^{-1}F\left(Y(t)\right)\right)_i &\geqslant a\frac{(Y_i(t))^p}{Y_{n-1,i}} - A_{ii}\left(1 + Z_{n,i}(s)\right),\\ \left(D_n^{-1}F\left(Y(t)\right)\right)_i &\geqslant \left(1 + Z_{n,i}(s)\right)\left(a\left(Y_i(t)\right)^{p-1} - A_{ii}\right).\end{aligned} \tag{27}$$

Hence:

$$\left\|D_n^{-1}F\left(Y(t)\right)\right\| \geqslant \left(D_n^{-1}F\left(Y(t)\right)\right)_i \geqslant \left(1 + Z_{n,i}(s)\right)\left(a\left(Y_i(t)\right)^{p-1} - A_{ii}\right).$$

For $i = i_0$ where $0 \leqslant Z_{n,i_0} \leqslant S$, one gets:

$$\left\|D_n^{-1}F\left(Y(t)\right)\right\| \geqslant a\left(Y_{i_0}(t)\right)^{p-1} - A_{i_0i_0},$$

yielding:

$$\begin{aligned}\frac{\|D_n^{-1}F(Y(t))\|}{\|Y(t)\|^{p-1}} &\geqslant a - \frac{a_0}{\|Y(t)\|^{p-1}},\\ \frac{\|D_n^{-1}F(Y(t))\|}{\|Y(t)\|^{p-1}} &\geqslant a - \frac{a_0}{\|Y(t)\|^{p-1}}.\end{aligned}$$

Then, there exists $\delta_2 > 0$ such that for n sufficiently large and $\forall t \in [T_{n-1}, T_n]$:

$$\frac{\|D_n^{-1}F(Y(t))\|}{\|Y(t)\|^{p-1}} \geqslant \delta_2 = constant. \tag{28}$$

On the other hand, one has:

$$D_n^{-1}F\left(Y(t)\right) = -D_n^{-1}AY(t) + D_n^{-1}\phi\left(Y(t)\right) = -D_n^{-1}AD_n\left(\mathbf{e} + Z_n(s)\right) + a\left(D\left(Y(t)\right)\right)^{p-1}\left(\mathbf{e} + Z_n(s)\right).$$

It follows:

$$\left\|D_n^{-1}F\left(Y(t)\right)\right\| \leqslant \left\|D_n^{-1}AD_n\right\|(1+S) + a\|Y_{n-1}\|^{p-1}(1+S)^p,$$

and therefore:

$$\frac{\|D_n^{-1}F(Y(t))\|}{\|Y_{n-1}\|^{p-1}} \leqslant \frac{\|D_n^{-1}AD_n\|}{\|Y_{n-1}\|^{p-1}}(1+S) + a(1+S)^p. \tag{29}$$

A first estimate on $\|D_n^{-1}AD_n\|$ is given by:

$$\left\|D_n^{-1}AD_n\right\| \leqslant \left\|D_n^{-1}\right\| \times \|A\| \times \|D_n\| = \frac{1}{\min_i(Y_{n-1,i})}\|A\| \times \|D_n\|,$$

implying, since $\min_i(Y_{n-1,i}) \geqslant \gamma_0$:

$$\left\|D_n^{-1}AD_n\right\| \leqslant \frac{1}{\gamma_0}\|A\| \times \|D_n\|.$$

Therefore, (29) yields:

$$\frac{\|D_n^{-1}F(Y(t))\|}{\|Y_{n-1}\|^{p-1}} \leqslant \frac{1}{\gamma_0}\|A\| \times \|Y_{n-1}\|^{2-p}(1+S) + a(1+S)^p, \tag{30}$$

so that if $p \geqslant 2$, there exists $\delta_3 > 0$ such that for n sufficiently large and $\forall t \in [T_{n-1}, T_n]$:

$$\frac{\|D_n^{-1}F(Y(t))\|}{\|Y_{n-1}\|^{p-1}} \leqslant \delta_3 = constant. \tag{31}$$

In the case $1 \leqslant p < 2$, a sharper estimate on $\|D_n^{-1}AD_n\|$ is needed. For that purpose, we start with:

$$\left\| D_n^{-1} A D_n \right\| = \max_i \left\{ A_{ii} + \sum_{j \in N(i)} |A_{ij}| \frac{Y_{n-1,j}}{Y_{n-1,i}} \right\}.$$

Using the diagonal dominance of A yields:

$$\left\| D_n^{-1} A D_n \right\| \leqslant \max_i \left\{ A_{ii} \left(1 + \max_{j \in N(i)} \frac{Y_{n-1,j}}{Y_{n-1,i}} \right) \right\},$$

and therefore:

$$\left\| D_n^{-1} A D_n \right\| \leqslant a_0 \left(1 + \max_{j \in N(i)} \frac{Y_{n-1,j}}{Y_{n-1,i}} \right).$$

So if the solution $Y(t)$ of (16) satisfies the neighbourhood condition:

$$\max_{i, j \in N(i)} \frac{Y_{n-1,j}}{Y_{n-1,i}} \quad \text{is } \mathrm{O}\left(\|Y_{n-1}\|^{p-1} \right), \tag{32}$$

then, for $1 \leqslant p < 2$ there exists $\delta_4 > 0$ such that for n sufficiently large and $\forall t \in [T_{n-1}, T_n]$:

$$\frac{\|D_n^{-1} F(Y(t))\|}{\|Y_{n-1}\|^{p-1}} \leqslant \delta_4 = constant. \tag{33}$$

It follows from (31) and (33) that if $p \geqslant 2$, or if $1 \leqslant p < 2$ with the neighbourhood condition (32) satisfied, then one has:

$$\left\| D_n^{-1} F\left(Y(t)\right) \right\| \quad \text{is } \mathrm{O}\left(\left\| Y(t) \right\|^{p-1} \right).$$

Together with (28), this implies that $\|D_n^{-1} F(Y(t))\|$ is $\Theta(\|Y(t)\|^{p-1})$.

Note that the neighbourhood condition (32) has been numerically tested in the experiments conducted at the end of this paper.

2.4. Validity of assumption (17.3)

Note first that $D_n^{-1} F_Y(Y(t)) D_n = -D_n^{-1} A D_n + ap(D(Y(t)))^{p-1}$. Hence: $\|D_n^{-1} F_Y(Y(t)) D_n\| \leqslant \|D_n^{-1} A D_n\| + ap\|Y(t)\|^{p-1}$, implying:

$$\frac{\|D_n^{-1} F_Y(Y(t)) D_n\|}{\|Y(t)\|^{p-1}} \leqslant \frac{\|D_n^{-1} A D_n\|}{\|Y(t)\|^{p-1}} + ap. \tag{34}$$

In the same way than above, one deduces that $\|D_n^{-1} F_Y(Y(t)) D_n\|$ is $\mathrm{O}(\|Y(t)\|^{p-1})$ if $p \geqslant 2$, or if $1 \leqslant p < 2$ with the neighbourhood condition (32) satisfied.

2.5. Validity of assumption (18)

Using inequality (27) at $i = i_0$ and assumption (23) one gets:

$$\left(D_n^{-1} F\left(Y(t)\right) \right)_{i_0} \geqslant \left(1 + Z_{n,i_0}(s)\right) \left(a\left(Y_{i_0}(t)\right)^{p-1} - A_{i_0 i_0} \right) \geqslant a\left(Y_{i_0}(t)\right)^{p-1} - a_0.$$

Hence:

$$\frac{(D_n^{-1} F(Y(t)))_{i_0}}{\|Y(t)\|^{p-1}} \geqslant a - \frac{a_0}{\|Y(t)\|^{p-1}}.$$

For large n, one has $\frac{1}{\|Y(t)\|^{p-1}} \to 0$ and therefore there exists a constant c, independent of n, such that for all $t \in [T_{n-1}, T_n]$:

$$\frac{(D_n^{-1} F(Y(t)))_{i_0}}{\|Y(t)\|^{p-1}} \geqslant c.$$

3. Local and global explicit numerical solvers

Uniform similarity allows for "similar numerical simulations" on all time slices. This renders this approach a **globally adaptive** numerical procedure. The control imposed on the growth of $\|Z_n(s)\|$ and the uniformity imposed on the governing function $G_n(.)$ allows to choose an accurate high order explicit numerical scheme to solve (12). In this way, one avoids major computational issues resulting from solving directly extremely stiff and exploding systems such as (6).

3.1. Local explicit solver

Once all slices are made uniformly similar, an explicit solver `ExplicitSolveZ`, can be used to discretize (12). In our case, we have implemented a standard **explicit Runge–Kutta method of order** 4. Specifically, the solver can be put in the following framework:

1. The solver is based on a routine that selects the initial time step τ_0, using ϵ_{tol}, as a tolerance for such choice.
2. After selecting τ_0, the discrete solution $\{z_n^j \mid j = 0, 1, \ldots, l, l+1\}$ is advanced within a slice through the algorithm:

$$[z_n, s] = \texttt{ExplicitSolveZ}(G_n):$$

$$\begin{cases} z_n^0 = 0 & (35.1) \\ z_n^j = \mathrm{RK4}\big(G_n, z_n^{j-1}\big), \quad j = 1, \ldots, l+1 & (35.2) \\ \big\|z_n^j\big\| < S, \quad 0 < j < l \quad \text{and} & \\ \big\|z_n^{l+1}\big\| > S, \quad s^{l+1} - s^l \leqslant \epsilon_{tol} & (35.3) \end{cases} \tag{35}$$

(35.3) is a bracketing procedure meant to allow the discrete solution $\{z_n\}$ to reach the end-of-slice condition $H(Z_n) = \|Z_n\| - S = 0$ within the required tolerance.

Therefore, within a rescaled nth slice, the algorithm `ExplicitSolveZ(`G_n`)` generates:

– a slice dependent sequence of rescaled times: $\{s^j \mid j = 0, 1, \ldots, l, l+1\}$ with:

$$0 = s^0 < s^1 < \cdots < s^{l-1} < s^l \leqslant s_n < s^{l+1} \quad \big(s^l \approx s_n\big).$$

– $\{z_n^j \mid j = 0, 1, \ldots, l\}$ the sequence of rescaled approximations to: $\{Z_n(s^j) \mid j = 0, 1, \ldots, l\}$,

with $\{s_n, Z_n(.)\}$ solving the continuous rescaled system (12). Thus applying (35) on (12), one reaches the following estimates:

$$\begin{cases} s^{l+1} - s^l \leqslant \epsilon_{tol}, \quad s^l \leqslant s_n < s^{l+1} & (36.1) \\ \max_{1 \leqslant j \leqslant l}\big\{\big\|Z_n\big(s^j\big) - z_n^j\big\|\big\} \leqslant \epsilon_{loc} & (36.2) \\ 0 \leqslant s_n - s^l \leqslant \epsilon_{tol} \quad \text{and} \quad \big\|Z_n(s_n) - z_n^l\big\| \leqslant \epsilon_{loc} & (36.3) \end{cases} \tag{36}$$

where ϵ_{loc} is the resulting maximum absolute error on all rescaled slices, with $\epsilon_{loc} = c\epsilon_{tol}$. Note here that as $S = O(1)$, absolute error estimates are equivalent to relative error estimates on $Z_n(s_n) - z_n^l$. Such is not the case if one is computing directly the solution $Y(t)$ of (6). Evidently, in such a case, the magnitude of $\|Y(t)\|$ would then impose use of relative errors.

Consequently, assuming that the initial value Y_{n-1} is exact, i.e. $Y_{n-1} = Y(T_{n-1})$ at the onset of an nth slice and once z_n is obtained using the local solver (35), one "reconstitutes" approximations y_n to the solution $Y(t)$ of (6) by letting:

1. $s_n \approx s_n^a = s^l$ (end-slice value of the rescaled time variable s at the nth slice),
2. $t^j = T_{n-1} + \beta_n s^j$, $j = 1, \ldots, l$ and $y_n^j = D_n(\mathbf{e} + z_n^j) \approx Y(t^j) = D_n(\mathbf{e} + Z_n(s^j))$, $j = 1, \ldots, l$.

Then, on the basis of solver (35) that provides approximation to $Z_n(s)$ on the nth slice, we construct the routine:

$$[s_n^a, z_n^a, Y_n^a, T_n^a] = \texttt{ExplicitSolveY}(Y_{n-1}, T_{n-1}, F, S, \epsilon_{tol})$$

Step 1: Determine β_n and $G_n(.)$.
Step 2: Apply `ExplicitSolveZ`(G_n). Obtain $\{z_n\}$ and the discrete mesh $\{s^j \mid j = 0, 1, \ldots, l\}$.
Step 3: Let $s_n^a = s^l$ and $z_n^a = z_n^l$.
Step 4: Reconstitute $\{y_n^j = D_n(\mathbf{e} + z_n^j) \mid j = 1, \ldots, l\}$ and $\{t^j = T_{n-1} + \beta_n s^j \mid j = 1, \ldots, l\}$.
Step 5: Obtain $Y_n^a = D_n(\mathbf{e} + z_n^a)$ and $T_n^a = T_{n-1} + \beta_n s^l$.

Thus, $s_n^a \approx s_n$, $T_n^a \approx T_n$, $z_n^a \approx Z_n(s_n)$ and $Y_n^a \approx Y_n = Y(T_n)$. These errors are governed by the local error generated by the algorithm `ExplicitSolveZ`, stated as follows:

Proposition 1. *If the rescaled systems* (12) *satisfy the properties of Uniform Similarity in Definition* 1*, one has*:

1. $\frac{\max_j \{\|Y(t^j)-y_n^j\|\}}{\|Y_n\|} \leqslant \max_j \{\|Z_n(s^j)-z_n^j\|\} \leqslant \epsilon_{loc}$ (*implying* $\frac{\|Y_n-Y_n^a\|}{\|Y_n\|} \leqslant \epsilon_{loc}$),
2. $0 \leqslant \frac{T_n-T_n^a}{T_n} \leqslant c\epsilon_{loc}$.

Proof. From:

$$y_n^j = D_n\big(\mathbf{e}+z_n^j\big) \approx Y\big(t^j\big) = D_n\big(\mathbf{e}+Z_n\big(s^j\big)\big), \quad j=1,\ldots,l,$$

one has:

$$Y\big(t^j\big) - y_n^j = D_n\big(Z_n\big(s^j\big)-z_n^j\big), \quad j=1,\ldots,l,$$

and therefore:

$$\max_j\big\{\big\|Y\big(t^j\big)-y_n^j\big\|\big\} \leqslant \Big(\max_j\big\{\big\|Z_n\big(s^j\big)-z_n^j\big\|\big\}\Big)\|Y_{n-1}\|,$$

so that:

$$\frac{\max_j \{\|Y(t^j)-y_n^j\|\}}{\|Y_n\|} \leqslant \max_j\big\{\big\|Z_n\big(s^j\big)-z_n^j\big\|\big\} \leqslant \epsilon_{loc}.$$

and consequently:

$$\frac{\|Y_n-Y_n^a\|}{\|Y_n\|} \leqslant \epsilon_{loc}.$$

On the other hand, using uniform similarity $c_0 \leqslant s_n$, one has:

$$0 \leqslant \frac{T_n-T_n^a}{T_n} = \frac{\beta_n}{T_n}\big(s_n-s_n^a\big) \leqslant \frac{\epsilon_{loc}}{c_0}. \qquad \square$$

3.2. Determining the total number of slices N

On the basis of the machine capacity being used, whereas `realmax` is the maximum number that can be reached and also given that any algorithm must evaluate $F(.)$, which satisfies (17.1), one has from the end-of-slice condition:

$$\forall n,\ \forall i, \quad \frac{|Y_i(t)-Y_{n-1,i}|}{Y_{n-1,i}} \leqslant S,$$

and therefore:

$$\forall n,\ \forall i, \quad \frac{Y_i(t)}{Y_{n-1,i}} \leqslant 1+\frac{|Y_i(t)-Y_{n-1,i}|}{Y_{n-1,i}} \leqslant 1+S.$$

This leads to:

$$\|Y_n\| \leqslant (1+S)\|Y_{n-1}\| \leqslant (1+S)^n\|Y_0\|.$$

Thus one obtains the following estimate on the maximum number of slices.

Theorem 2. *If* $F(.)$ *satisfies* (17.1)*, then the maximum number of slices satisfies* $N \leqslant N_{\max}$ *where*:

$$N_{\max} = \lfloor M \rfloor,$$

where:

$$M = \frac{\ln(\frac{\text{realmax}}{c\|Y_0\|^\gamma})}{\ln(1+S)^\gamma}.$$

Proof. It follows from use of the inequalities:

$$\big((1+S)^{N_{\max}}\|Y_0\|\big)^\gamma \leqslant \frac{\text{realmax}}{c} < \big((1+S)^{N_{\max}+1}\|Y_0\|\big)^\gamma. \qquad \square$$

Note that the number of slices decreases with S increasing.

3.3. Resulting globally explicit adaptive solver

On the basis of a slice solver `ExplicitSolveY` based on an ϵ_{tol} accuracy, we may then define a Globally Adaptive solver for (1) that would generate an approximation to $Y(t)$ in the following way:

$[\{s_n^c\}, \{z_n^c\}, \{Y_n^c\}, \{T_n^c\}] = \texttt{GlobalExplicitAdaptiveSolver}(Y_0, T_0 = 0, F, S, \epsilon_{tol})$

Base step: $[s_1^c, z_1^c, Y_1^c, T_1^c] = \texttt{ExplicitSolveY}(Y_0, T_0 = 0, F, S, \epsilon_{tol})$,
yielding an approximation to $Y(t)$ on a first slice and at its end, the pair $\{Y_1^c, T_1^c\}$,
with: $\frac{|T_1 - T_1^c|}{T_1} \leqslant c\epsilon_{loc}$ and $\frac{\|Y_1 - Y_1^c\|}{\|Y_1\|} \leqslant \epsilon_{loc}$.

Inductive step: $[s_n^c, z_n^c, Y_n^c, T_n^c] = \texttt{ExplicitSolveY}(Y_{n-1}^c, T_{n-1}^c, F, S, \epsilon_{tol})$,
reaching an approximation $\{T_n^c, Y_n^c\}$ of $\{T_n,\ Y(T_n)\}$, at the end of the nth slice,
within the tolerance: $\epsilon_n = \max\{\frac{|T_n - T_n^c|}{T_n}, \frac{\|Y_n - Y_n^c\|}{\|Y_n\|}\}$.

Thus, unlike the first slice where one starts the slice with the exact initial value Y_0, the starting value on the nth slice $(n > 1)$ is $Y_{n-1}^c \approx Y_{n-1}$. In such context, the governing rescaled system on the nth slice, $n > 1$, is not (12), but:

$$\begin{cases} \dfrac{dZ_n^c}{ds} = G_n^c\big(Z_n^c\big), \quad 0 < s \leqslant \overline{s_n^c} & (37.1) \\ Z_n^c(0) = 0 & (37.2) \\ \big\|Z_n^c\big(\overline{s_n^c}\big)\big\| = S \quad \text{and} \quad \big\|\big(Z_n^c(s)\big)\| < S, \quad 0 < s < \overline{s_n^c} & (37.3) \end{cases} \tag{37}$$

$$\text{with } G_n^c(W) = \beta_n^c F\big(D_n^c(\mathbf{e} + W)\big); \qquad \beta_n^c = \frac{1}{\|D_n^c F(Y_{n-1}^c)\|},$$

and D_n^c the diagonal matrix associated with Y_{n-1}^c.

3.4. Stability assumption

Letting:

$$\overline{s_n} = \max\big\{s_n, \overline{s_n^c}\big\},$$

and extending $Z_n(.)$ and $Z_n^c(.)$ on $[0, \overline{s_n}]$, we make the following assumptions on (12) and (37):

For $S > 0$, there exists a constant $K(S)$ independent from n, such that the solutions of (12) and (37) satisfy the **stability property**:

$$\begin{cases} \max_{s \leqslant \overline{s_n}} \big\|Z_n(s) - Z_n^c(s)\big\| \leqslant K\epsilon_{n-1} & (38.1) \\ \big|s_n - \overline{s_n^c}\big| \leqslant K\epsilon_{n-1} & (38.2) \end{cases} \tag{38}$$

with K a generic constant independent from n and depending on S and $F(.)$. Letting:

$$\epsilon_{glob} = \max_{1 \leqslant n \leqslant N} \epsilon_n,$$

we then prove in Section 4:

Theorem 3. *Let N be the total number of slices. Then under the assumptions of Theorem 1 and the stability assumption (38), one has*:

$$\max_{1 \leqslant n \leqslant N} \left\{ \frac{\|Y_n - Y_n^c\|}{\|Y_n\|} \right\} \leqslant \epsilon_{glob} = \epsilon_N \leqslant (1 + S + K)^N c\epsilon_{loc}.$$

3.5. Steps for validating the stability property (38)

Verifying (38) starts by proving first that:

$$\forall W \in \mathbb{R}^k,\ \|W\| \leqslant S,\ \text{one has:} \quad \big\|G_n(W) - G_n^c(W)\big\| \leqslant K\epsilon_{n-1}. \tag{39}$$

Once such estimate obtained, (38) is then proved by using the integral equation:

$$Z_n(s) - Z_n^c(s) = \int_0^s \left(G_n\big(Z_n(s)\big) - G_n^c\big(Z_n^c(s)\big)\right) ds.$$

Appropriate estimates derived from this last identity using (39) and the uniform property on $G_n(.)$ (parts 1 and 2 of Definition 1) would then lead to (38).

As for verifying (39), one starts with the identity $D_n^c = D_n(I + \Lambda_n)$, I being the identity matrix and Λ_n a diagonal matrix such that $\|\Lambda_n\| = \epsilon_{n-1}$. This is followed by obtaining an expression of $G_n(W) - G_n^c(W)$ that can be estimated in terms of Λ_n. Such validation have been carried for the cases of reaction–diffusion problem on which we have conducted numerical tests in Section 5.

4. Proof of Theorem 3

Consider then the sequence $\{(T_n^c, Y_n^c) \mid n = 1, 2, \ldots, N\}$ obtained from the global solver defined in (3.3). One has $T_0^c = 0$, $Y_0^c = Y_0$ and from Theorem 1:

$$\frac{\|Y_1^c - Y_1\|}{\|Y_1\|} \leqslant \epsilon_1 = c\epsilon_{loc} \quad \text{and} \quad \frac{|T_1 - T_1^c|}{T_1} \leqslant \epsilon_1 = c\epsilon_{loc}.$$

On the other hand, for $n \geqslant 1$, the pair (T_n^c, Y_n^c) is given through the function:

$$\big[s_n^c, z_n^c, Y_n^c, T_n^c\big] = \texttt{ExplicitSolveY}\big(Y_{n-1}^c, T_{n-1}^c, F, S, \epsilon_{tol}\big),$$

so that:

$$T_n^c = T_{n-1}^c + \beta_n^c s_n^c; \qquad Y_n^c = D_n^c\big(\mathbf{e} + z_n^c\big),$$

with the pair (T_{n-1}, Y_{n-1}) satisfying:

$$\frac{\|Y_{n-1}^c - Y_{n-1}\|}{\|Y_{n-1}\|} \leqslant \epsilon_{n-1} \quad \text{and} \quad \frac{|T_{n-1} - T_{n-1}^c|}{T_{n-1}} \leqslant \epsilon_{n-1}.$$

To obtain estimates on $\frac{\|Y_n^c - Y_n\|}{\|Y_n\|}$ and $\frac{|T_n - T_n^c|}{T_n}$ and therefore a relationship between ϵ_n and ϵ_{n-1}, we start with:

$$Y_n^c = D_n^c\big(e + z_n^c\big) \approx Y_n = D_n\big(e + Z_n(s_n)\big),$$

and obtain the identity:

$$Y_n - Y_n^c = Y_{n-1} - Y_{n-1}^c + D_n Z_n(s_n) - D_n^c z_n^c,$$

and consequently:

$$Y_n - Y_n^c = \big(Y_{n-1} - Y_{n-1}^c\big) + D_n\big(Z_n(s_n) - Z_n^c\big(\overline{s_n^c}\big)\big) + D_n\big(Z_n^c\big(\overline{s_n^c}\big) - z_n^c\big) + \big(D_n - D_n^c\big)z_n^c,$$

where $Z_n^c(.)$ solves (37). Using triangle inequality, one obtains:

$$\big\|Y_n - Y_n^c\big\| \leqslant \big\|Y_{n-1} - Y_{n-1}^c\big\| + \|D_n\|.\big\|Z_n(s_n) - Z_n^c\big(\overline{s_n^c}\big)\big\| + \|D_n\|.\big\|\big(Z_n^c\big(\overline{s_n^c}\big) - z_n^c\big)\big\| + \big\|D_n - D_n^c\big\|.\big\|z_n^c\big\|. \tag{40}$$

Note that the first and last terms on the right-hand side of (40) are easily estimated:

$$\big\|Y_{n-1} - Y_{n-1}^c\big\| \leqslant \epsilon_{n-1}\|Y_{n-1}\| \leqslant \epsilon_{n-1}\|Y_n\| \quad \text{and}$$
$$\big\|D_n - D_n^c\big\|.\big\|z_n^c\big\| \leqslant S\big\|Y_{n-1} - Y_{n-1}^c\big\| \leqslant S\epsilon_{n-1}\|Y_{n-1}\| \leqslant S\epsilon_{n-1}\|Y_n\|.$$

We focus now on the second and third terms on the right-hand side of (40). For the second term, we derive the following lemma:

Lemma 5. *Under assumption (38) and the uniform similarity properties of the rescaled systems (12) and (37), one has*:

$$\big\|Z_n(s_n) - Z_n^c\big(\overline{s_n^c}\big)\big\| \leqslant K\epsilon_{n-1}. \tag{41}$$

K generic constant independent form n.

Proof. As both solutions to (12) and (37) are extended on $[0, \overline{s_n}]$, note then that:

$$Z_n(s_n) - Z_n^c\big(\overline{s_n^c}\big) = \big(Z_n(s_n) - Z_n\big(\overline{s_n^c}\big)\big) + \big(Z_n\big(\overline{s_n^c}\big) - Z_n^c\big(\overline{s_n^c}\big)\big).$$

The following results from the uniform boundedness $G_n(.)$ and from (38.2):

$$\left\| Z_n(s_n) - Z_n\left(\overline{s_n^c}\right) \right\| \leqslant C \left| s_n - \overline{s_n^c} \right| \leqslant K\epsilon_{n-1},$$

and based on (38.1), one has:

$$\left\| Z_n\left(\overline{s_n^c}\right) - Z_n^c\left(\overline{s_n^c}\right) \right\| \leqslant K\epsilon_{n-1}. \qquad \square$$

It follows from this lemma that: $\|D_n\|.\|Z_n(s_n) - Z_n^c(\overline{s_n^c})\| \leqslant K\epsilon_{n-1}\|Y_n\|$.

For the third term on the right-hand side of (40), note that when applying the local solver `ExplicitSolveZ`, one gets: $\|Z_n^c(\overline{s_n^c}) - z_n^c\| \leqslant \epsilon_{loc}$ and $|\overline{s_n^c} - s_n^c| \leqslant c\epsilon_{loc}$, implying similarly:

$$\|D_n\|.\left\| \left(Z_n^c\left(\overline{s_n^c}\right) - z_n^c\right) \right\| \leqslant K\epsilon_{n-1}\|Y_n\|. \tag{42}$$

Thus, one concludes from (40), (42) and Lemma 5 that:

$$\frac{\|Y_n - Y_n^c\|}{\|Y_n\|} \leqslant \epsilon_{n-1} + c\epsilon_{loc} + (K+S)\epsilon_{n-1},$$

and gets the recurrence relation on the sequence $\{\epsilon_n\}$:

$$\epsilon_n = c\epsilon_{loc} + (1 + S + K)\epsilon_{n-1}, \tag{43}$$

then by solving (43) with $\epsilon_1 = c\epsilon_{loc}$, one achieves the proof of Theorem 3.

Remark. The theoretical estimate of Theorem 3 indicates that:

$$\frac{\epsilon_{glob}}{\epsilon_{loc}} = (1 + S + K)^N$$

varies exponentially with N. This means that the method is convenient if its implementation is done on a few number of slices, as for example in the case of a finite existence (blow-up) time. However, even for problems with long existence time, numerical experiments we have conducted indicate that:

$$1 < \left(\frac{\epsilon_{glob}}{\epsilon_{loc}} \right)^{\frac{1}{N}} < 1 + S,$$

which imply that the estimate of Theorem 3 is not a sharp one.

5. Numerical experiments

Two explosive cases are being considered: (i) a linear infinite time existence problem and (ii) a semi-linear finite time blow-up one.

5.1. Linear reaction–diffusion problem

Obtained from (13) when $m = p = 1$:

$$\frac{\partial u}{\partial t} = \Delta u + au, \quad x \in \Omega, \qquad u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \qquad u(x,0) = u_0(x) > 0. \tag{44}$$

A space semi-discretization of dimension k, yields then the equivalent linear initial value problem:

$$\frac{dY}{dt} = BY, \quad t > 0, \qquad Y(0) = Y_0, \tag{45}$$

where $B = aI - A \in \mathbb{R}^{k\times k}$, A being a sparse symmetric positive definite matrix that discretizes the operator $-\Delta$ and I the identity matrix. Assuming that $a > \lambda_1$, where λ_1 is the smaller eigenvalue of $-\Delta$, the solution of this problem is explosive in infinite time and is analytically known to be: $Y(t) = \exp(Bt)Y_0$. Using the EOS condition (9.3) and rescaling with the critical choice (19) for β_n make solving problem (5.1) equivalent to solving, on each nth slice $[0, s_n]$, corresponding to $[T_{n-1}, T_n]$, a linear initial value shooting problem:

$$\begin{cases} \dfrac{dZ_n}{ds} = B_n(\mathbf{e} + Z_n) = G_n(Z_n), \quad 0 < s < s_n & (46.1) \\ Z_n(0) = 0, & (46.2) \\ \forall s < s_n, \quad \left\| Z_n(s) \right\|_\infty < S, \quad \text{and} \quad \left\| Z_n(s_n) \right\|_\infty = S. & (46.3) \end{cases} \tag{46}$$

where $B_n = \beta_n D_n^{-1} B D_n$ is a matrix in $\mathbb{R}^{k\times k}$, constant on each nth slice, since D_n depends only on the starting value Y_{n-1} at the nth slice. The analytic solution of the rescaled linear problem (46), on each nth slice is: $Z_n(s) = (\exp(B_n s) - I)\mathbf{e}$. Because of the availability of an exact analytic solution, one can evaluate, at each nth slice the local and global errors, respectively:

Table 1
Linear reaction–diffusion problem – linear case – local and global errors over the first 100 slices.

ε_{tol}	1.E-09	1.E-07	1.E-05
τ_0	4.8828125E-04	1.9531250E-03	3.9062500E-03
T_{10}	3.3090710E+01	3.3090708E+01	3.3090518E+01
ε_{loc}	7.8159701E-15	1.9788618E-13	4.0911835E-12
ε_{glob}	3.0509020E-13	7.6284939E-13	3.0483840E-11
$\left(\varepsilon_{glob}/\varepsilon_{loc}\right)^{1/10}$	1.4425954E+00	1.1444645E+00	1.2224246E+00
T_{20}	6.6239772E+01	6.6239768E+01	6.6239369E+01
ε_{loc}	6.3948846E-14	1.7426063E-13	4.1551332E-12
ε_{glob}	2.6901187E-12	4.0059881E-12	6.7188834E-11
$\left(\varepsilon_{glob}/\varepsilon_{loc}\right)^{1/20}$	1.4534292E+00	1.3682046E+00	1.3209039E+00
T_{30}	9.9388834E+01	9.9388828E+01	9.9388219E+01
ε_{loc}	4.7073456E-14	1.7585935E-13	4.2032732E-12
ε_{glob}	4.1511623E-13	8.5817745E-12	1.0025677E-10
$\left(\varepsilon_{glob}/\varepsilon_{loc}\right)^{1/30}$	1.2431953E+00	1.4751669E+00	1.3732595E+00
T_{50}	1.6568696E+02	1.6568695E+02	1.6568592E+02
ε_{loc}	8.8817842E-15	2.1778137E-13	4.1601071E-12
ε_{glob}	3.3129195E-12	1.3956394E-11	1.7524714E-10
$\left(\varepsilon_{glob}/\varepsilon_{loc}\right)^{1/50}$	1.8078861E+00	1.5159164E+00	1.4536326E+00
T_{100}	3.3143227E+02	3.3143225E+02	3.3143017E+02
ε_{loc}	4.9205084E-14	1.9273474E-13	4.2425312E-12
ε_{glob}	9.6180758E-12	1.7023807E-11	3.4230211E-10
$\left(\varepsilon_{glob}/\varepsilon_{loc}\right)^{1/100}$	1.6947585E+00	1.5653435E+00	1.5512381E+00

- $\epsilon_{loc} = \max_l \|Z_n(s^l) - z_n^c\|_\infty$,
- $\epsilon_{glob} = \frac{\|Y_n - Y_n^c\|_\infty}{\|Y_n\|_\infty}$.

Table 1 provides the numerical results for the case:

$$a = 3, \quad S = 5, \quad u_0(x) = 1 - x^2 \quad \text{with } \Omega = [-1, 1] \subset \mathbb{R}.$$

Table 2
Semi-linear reaction–diffusion problem – blow-up in finite time – error on T_b.

ϵ_{tol}	1.E-05	1.E-07	1.E-09	1.E-11	1.E-14
τ_0	3.906250E-03	1.953125E-03	9.765625E-04	2.441406E-04	3.0517578E-05
Blow-up at slice	30	43	56	68	88
T_b^c	3.787826E+00	3.787862E+00	3.787863E+00	3.787863E+00	$T_b = 3.7878626$
Relative error on T_b^c	3.574101E-05	3.386474E-07	3.210220E-09	4.351985E-11	
$\max(s_n^c)$	1.505569E+00	1.505861E+00	1.505864E+00	1.505864E+00	$\max(s_n) = 1.5058644$
Maximum relative error on s_n^c	1.598845E-05	1.106711E-07	1.760410E-09	1.417999E-11	

Note that the results on the row that computes $(\frac{\epsilon_{glob}}{\epsilon_{loc}})^{1/N}$ indicates that the estimate proved in Theorem 3 is not a sharp one, as the value obtained is much less than $1 + S$.

5.2. Semi-linear reaction–diffusion problem, with finite time blow-up

When $m = 1$ and $p > 1$, the resulting semi-linear reaction–diffusion problem:

$$\frac{\partial u}{\partial t} = \Delta u + au^p, \quad p > 1,\ x \in \Omega, \qquad u(x,t) = 0, \quad x \in \partial\Omega,\ t \geqslant 0, \qquad u(x,0) = u_0(x) > 0, \tag{47}$$

is known to have a solution that is blowing up in finite time, at $T = T_b$.

The rescaling methodology has proved to be very efficient, in such cases, yielding a very accurate approximation of the finite time T_b of explosion (see [8] and [9]).

Since no analytic solution is available, the error on T_b cannot be calculated. However, when taking a very small computational tolerance ϵ_{tol} (10^{-14}, for example), one may consider the resulting value T_b most appropriate to use for the exact value.

The numerical results in Table 2 give the time of explosion T_b and the values of the local and global errors, when $p = 1.2$, $a = 3$, $S = 5$ and $u_0(x) = 1 - x^2$, with $\Omega = [-1, 1] \subset \mathbb{R}$.

Note in this case the extreme stability of the method, particularly the uniformity of the rescaled time variables that falls in the interval $[0, 1.5\ldots]$ Also, the algorithm appears to be robust in allowing to reach a blow-up time for very small computational tolerances.

6. Conclusions

Hence, we have shown that the use of an explicit numerical method to solve an explosive highly stiff problem is possible. However, this must be done within the framework of "sliced-time computations"; the explicit scheme being implemented on a sequence of time slices over which the original ODE system is transformed into uniformly similar systems. It appears clear that defining such rescaled systems is tied to an appropriate specification of the end-of-slice conditions. In the case of blowing-up solutions, keeping the rescaled solution $Z_n(.)$ within a ball $B \in \mathbb{R}^k = \{W:\ \|W\| \leqslant S\}$ is the most natural end the slice condition. Interestingly, the method can be applied for systems with finite or with infinite existence times. In the first case, the blow-up time is computed accurately demonstrating in this case, the efficiency of time-sliced computations.

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