Methods based on an energy formulation for data completion problems

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Inverse Problems, Computation, and Applications

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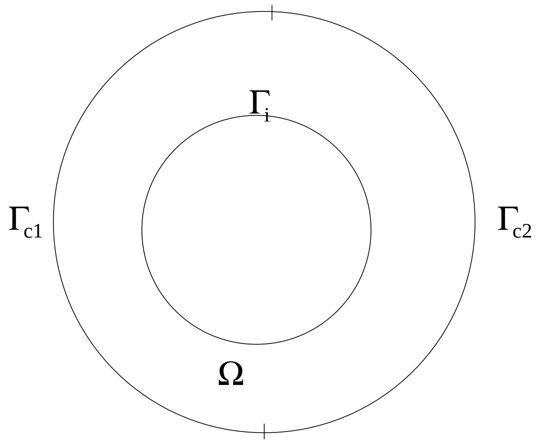
Outline

- Classical least-squares formulation
- Energy-like formulations
- First approach with Interface equations
- Second approach with Adjoint equations
- Numerical experiments

Data completion problem

Example of geometry



Model problem

$$\begin{cases} \nabla.\sigma\nabla u = 0 & \text{in} \quad \Omega, \\ \sigma\nabla u.n = \Psi & \text{on} \quad \Gamma_c, \\ u = \Phi & \text{on} \quad \Gamma_c. \end{cases}$$

Ill-posed problem with missing boundary data

One well-posed Dirichlet problem

Given Neumann condition and unknown Dirichlet condition

$$\begin{cases} \nabla.\sigma\nabla v = 0 & \text{in} \quad \Omega, \\ \sigma\nabla v.n = \Psi & \text{on} \quad \Gamma_c, \\ v = \eta & \text{on} \quad \Gamma_i. \end{cases}$$

Least-squares formulation

$$L(\eta) = \int_{\Gamma_c} (v - \Phi)^2 dx$$

problem equivalent to $\min_\eta L(\eta)$

One well-posed D problem

Splitting of solution

$$v = v_0 + r,$$

with

$$\begin{cases} \nabla.\sigma\nabla v_0 = 0 & \text{in} \quad \Omega, \\ \sigma\nabla v_0.n = \Psi & \text{on} \quad \Gamma_c, \\ v_0 = 0 & \text{on} \quad \Gamma_i. \end{cases}$$

and

$$\begin{cases} \nabla.\sigma\nabla r = 0 & \text{in} \quad \Omega, \\ \sigma\nabla r.n = 0 & \text{on} \quad \Gamma_c, \\ r = \eta & \text{on} \quad \Gamma_i. \end{cases}$$

One well-posed D problem

Transfer operator

$$
\begin{aligned}
&T : H^{1/2}(\Gamma_i) \rightarrow H^{1/2}(\Gamma_c),\\
&\eta \mapsto r.
\end{aligned}
$$

Least-squares equation

$$\min_{\eta} \|T\eta - \Phi - \Phi_0\|$$

rectangular ill-posed linear system : regularization required

Two well-posed D-D problems

Dirichlet-Dirichlet

$$\begin{cases} \nabla.\sigma\nabla v = 0 & \text{in} \quad \Omega, \\ \sigma\nabla v.n = \Psi & \text{on} \quad \Gamma_c, \\ v = \eta & \text{on} \quad \Gamma_i. \end{cases} \quad \begin{cases} \nabla.\sigma\nabla w = 0 & \text{in} \quad \Omega, \\ w = \Phi & \text{on} \quad \Gamma_c, \\ w = \eta & \text{on} \quad \Gamma_i. \end{cases}$$

Energy-like function

$$E(\eta) = \frac{1}{2}\int_\Omega \sigma(\nabla v - \nabla w)^2$$

problem equivalent to $\min_\eta E(\eta)$

First-order optimality conditions

$$\left\{ \sigma\nabla v.n - \sigma\nabla w.n = 0 \quad \text{on} \quad \Gamma_i. \right.$$

Two well-posed D-D problems

Splitting of solutions

$$v = v_0 + r, w = w_0 + s,$$

with

$$\begin{cases} \nabla.\sigma\nabla v_0 = 0 & \text{in} \quad \Omega, \\ \sigma\nabla v_0.n = \Psi & \text{on} \quad \Gamma_c, \\ v_0 = 0 & \text{on} \quad \Gamma_i. \end{cases} \begin{cases} \nabla.\sigma\nabla w_0 = 0 & \text{in} \quad \Omega, \\ w_0 = \Phi & \text{on} \quad \Gamma_c, \\ w_0 = 0 & \text{on} \quad \Gamma_i. \end{cases}$$

and

$$\begin{cases} \nabla.\sigma\nabla r = 0 & \text{in} \quad \Omega, \\ \sigma\nabla r.n = 0 & \text{on} \quad \Gamma_c, \\ r = \eta & \text{on} \quad \Gamma_i. \end{cases} \begin{cases} \nabla.\sigma\nabla s = 0 & \text{in} \quad \Omega, \\ s = 0 & \text{on} \quad \Gamma_c, \\ s = \eta & \text{on} \quad \Gamma_i. \end{cases}$$

Two well-posed D-D problems

Dirichlet to Neumann interface operators

$$R : H^{1/2}(\Gamma_i) \rightarrow H^{-1/2}(\Gamma_i), \quad S : H^{1/2}(\Gamma_i) \rightarrow H^{-1/2}(\Gamma_i),$$
$$\eta \mapsto \sigma \nabla r.n, \qquad \eta \mapsto \sigma \nabla s.n.$$

First-order optimality condition

$$(R - S)\eta = -\sigma \nabla v_0.n + \sigma \nabla w_0.n.$$

symmetric linear system

Two well-posed mixed D-N problems

Dirichlet-Neumann

$$\begin{cases} \nabla.\sigma\nabla v = 0 & \text{in} \quad \Omega, \\ \sigma\nabla v.n = \Psi & \text{on} \quad \Gamma_c, \\ v = \eta & \text{on} \quad \Gamma_i. \end{cases} \quad \begin{cases} \nabla.\sigma\nabla w = 0 & \text{in} \quad \Omega, \\ w = \Phi & \text{on} \quad \Gamma_c, \\ \sigma\nabla w = \tau & \text{on} \quad \Gamma_i. \end{cases}$$

Energy-like function

$$E(\eta,\tau) = \frac{1}{2}\int_\Omega \sigma(\nabla v - \nabla w)^2$$

problem equivalent to $\min_{\eta,\tau} E(\eta,\tau)$

First-order optimality conditions

$$\begin{cases} \sigma\nabla v.n - \sigma\nabla w.n = 0 & \text{on} \quad \Gamma_i, \\ v - w = \text{constant} & \text{on} \quad \Gamma_i. \end{cases}$$

Two well-posed mixed D-N problems

Splitting of solutions

$$v = v_0 + r, w = w_0 + s,$$

with

$$\begin{cases} \nabla.\sigma\nabla v_0 = 0 & \text{in} \quad \Omega, \\ \sigma\nabla v_0.n = \Psi & \text{on} \quad \Gamma_c, \\ v_0 = 0 & \text{on} \quad \Gamma_i. \end{cases} \begin{cases} \nabla.\sigma\nabla w_0 = 0 & \text{in} \quad \Omega, \\ w_0 = \Phi & \text{on} \quad \Gamma_c, \\ w_0 = 0 & \text{on} \quad \Gamma_i. \end{cases}$$

and

$$\begin{cases} \nabla.\sigma\nabla r = 0 & \text{in} \quad \Omega, \\ \sigma\nabla r.n = 0 & \text{on} \quad \Gamma_c, \\ r = \eta & \text{on} \quad \Gamma_i. \end{cases} \begin{cases} \nabla.\sigma\nabla s = 0 & \text{in} \quad \Omega, \\ s = 0 & \text{on} \quad \Gamma_c, \\ \sigma\nabla s.n = \tau & \text{on} \quad \Gamma_i. \end{cases}$$

Two well-posed mixed D-N problems

DtoN and NtoD interface operators

$$
\begin{array}{ll}
R : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i), & S^{-1} : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i), \\
\eta \mapsto \sigma \nabla r.n, & \tau \mapsto s
\end{array}
$$

First-order optimality conditions

$$
\begin{array}{l}
-R\eta + \tau = \sigma \nabla v_0.n - \sigma \nabla w_0.n, \\
\eta - S^{-1}\tau = w_0 - v_0.
\end{array}
$$

nonsymmetric linear system

Two well-posed mixed D-N problems

Lagrange multipliers and adjoint problems

$$\begin{cases} \nabla.\sigma\nabla\lambda = 0 & \text{in} \quad \Omega, \\ \sigma\nabla\lambda.n = \sigma\nabla v.n - \sigma\nabla w.n & \text{on} \quad \Gamma_c, \\ \lambda = 0 & \text{on} \quad \Gamma_i. \end{cases} \quad \begin{cases} \nabla.\sigma\nabla\mu = 0 & \text{in} \quad \Omega \\ \mu = 0 & \text{on} \quad \Gamma_c \\ \sigma\nabla\mu.n = \sigma\nabla w.n - \sigma\nabla v.n & \text{on} \quad \Gamma_i \end{cases}$$

Derivatives of energy function

$$\begin{cases} \frac{\partial E}{\partial\eta} = -\sigma\nabla\lambda.n - \sigma\nabla\mu.n & \text{on} \quad \Gamma_i, \\ \frac{\partial E}{\partial\tau} = \mu & \text{on} \quad \Gamma_i. \end{cases}$$

optimization problem $\min_{\eta,\tau} E$ using adjoint problems

Numerical experiment with D-D and interface problems

Synthetic data

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in} & \Omega \times [0,T], \\ \nabla u.n = G_1 & \text{on} & \Gamma_{c1} \times [0,T], \\ \nabla u.n = G_2 & \text{on} & \Gamma_{c2} \times [0,T], \\ u = F & \text{on} & \Gamma_i \times [0,T], \\ u(x,y,0) = 0 & \text{in} & \Omega \end{cases}$$

Smooth test case

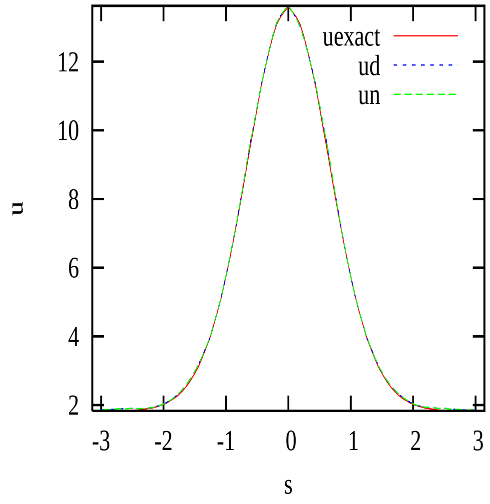
$F = te^x \cos y$ and $G_1 = G_2 = \nabla F.n$

Singular test case

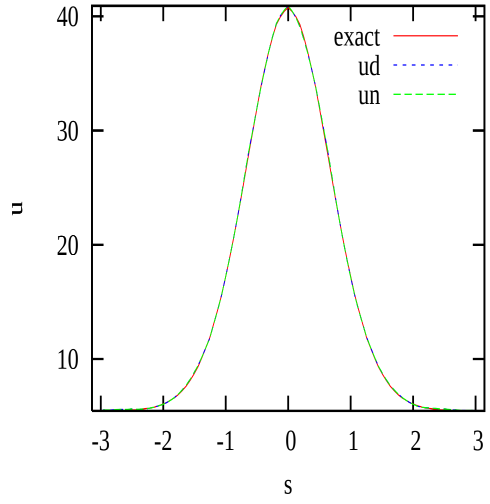
$F = Re(t/(z-a))$, $z = x + iy$, $G_1 = t$ and $G_2 = -t$ with $a = 0.8$

Numerical results for the smooth test case

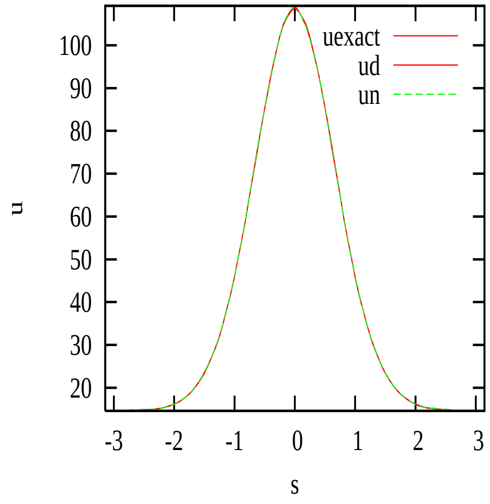
t=5



t=15

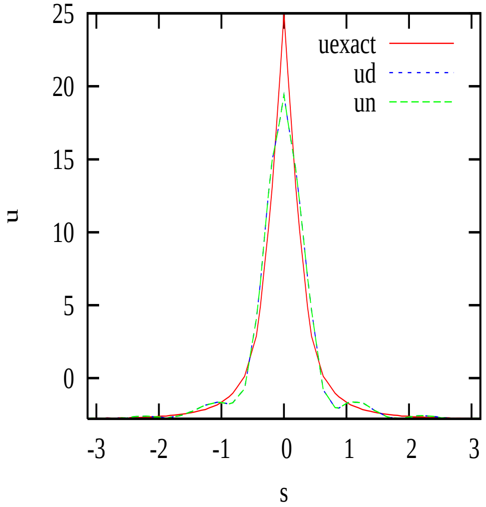


t=40

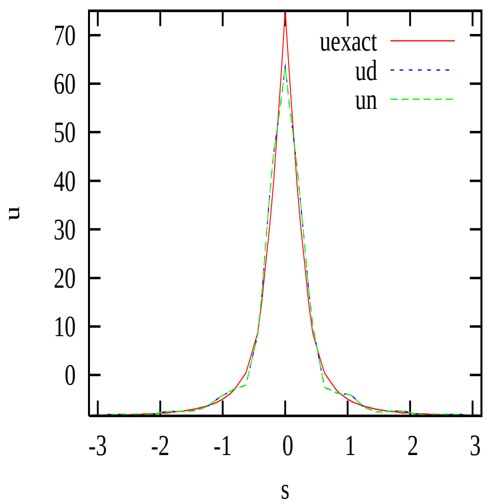


Numerical results for the singular test case

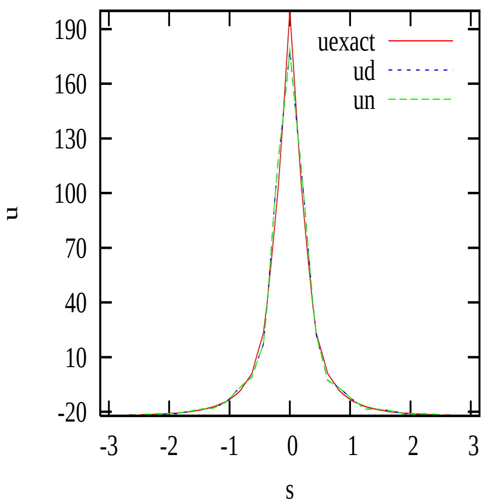
t=5



t=15

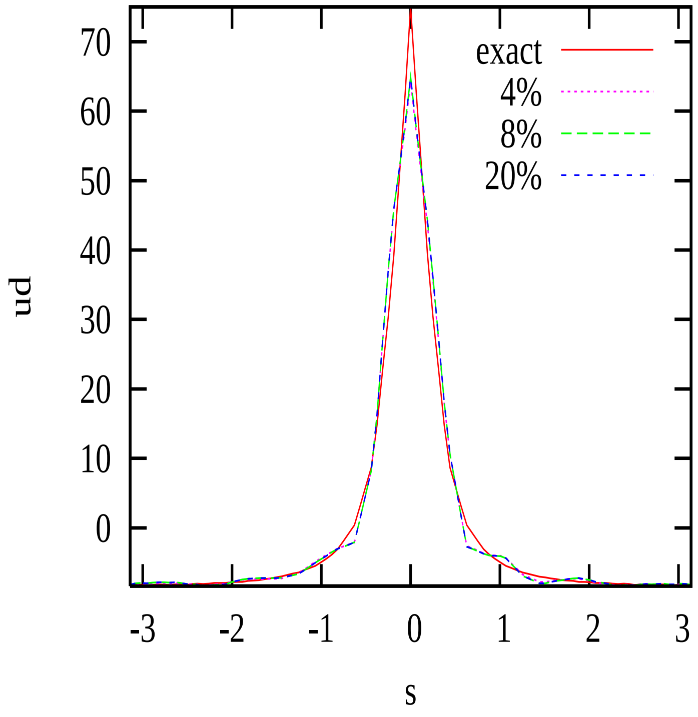


t=40

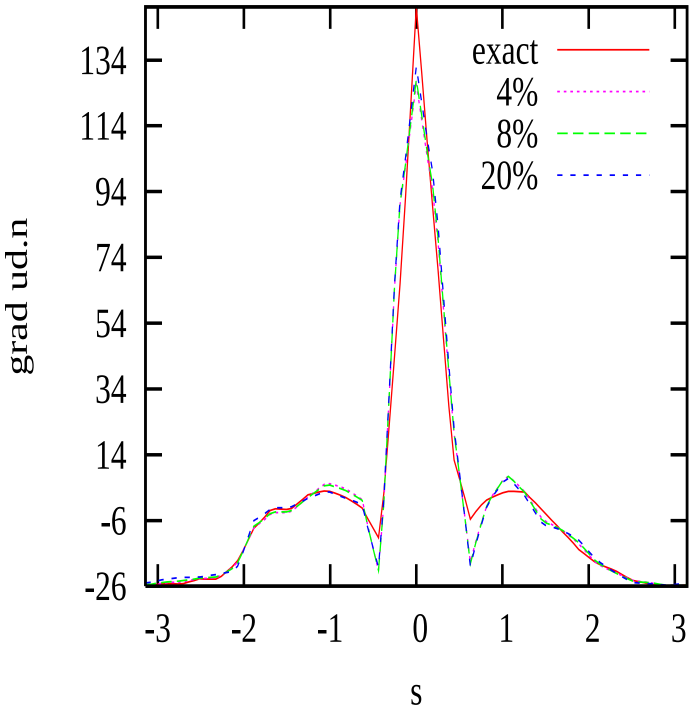


Numerical results for the noisy singular test case

t=15 u

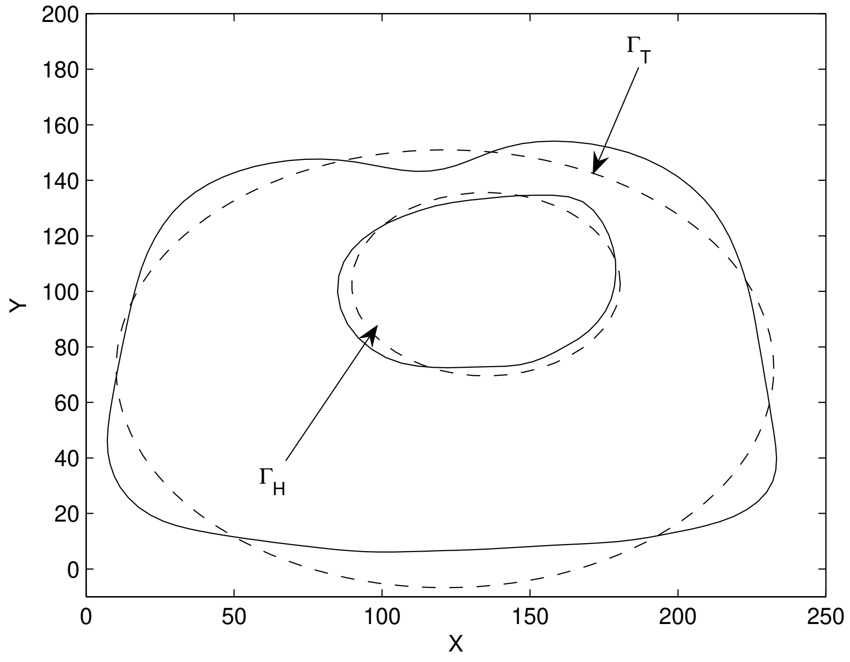


t=15 $\nabla u.n$



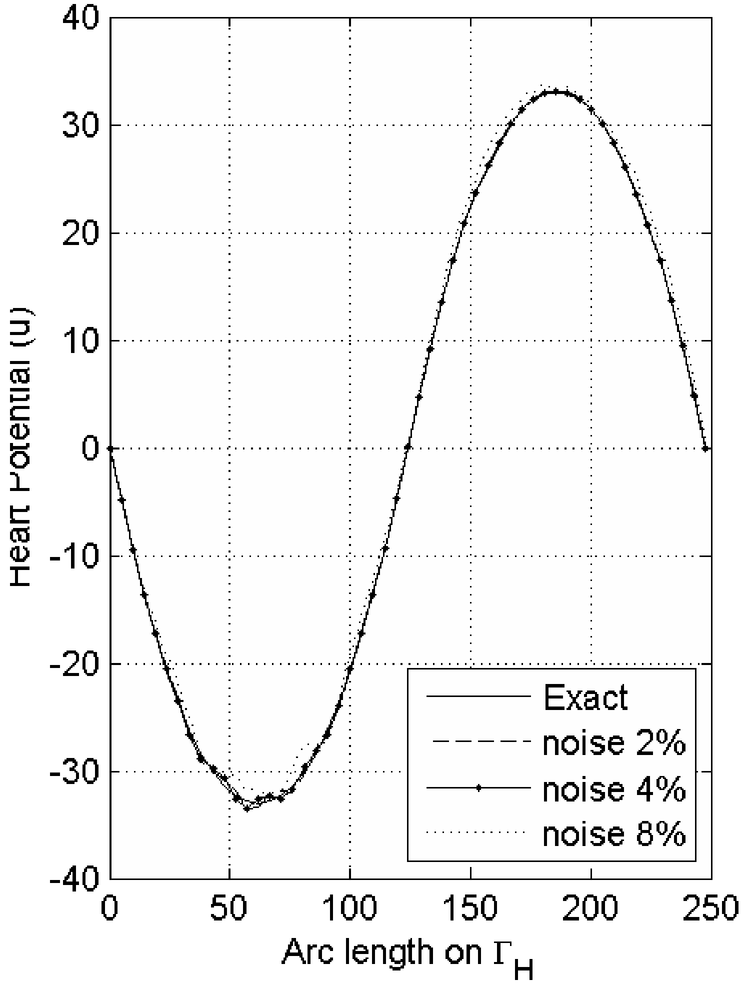
Numerical experiment with D-N and adjoint problems

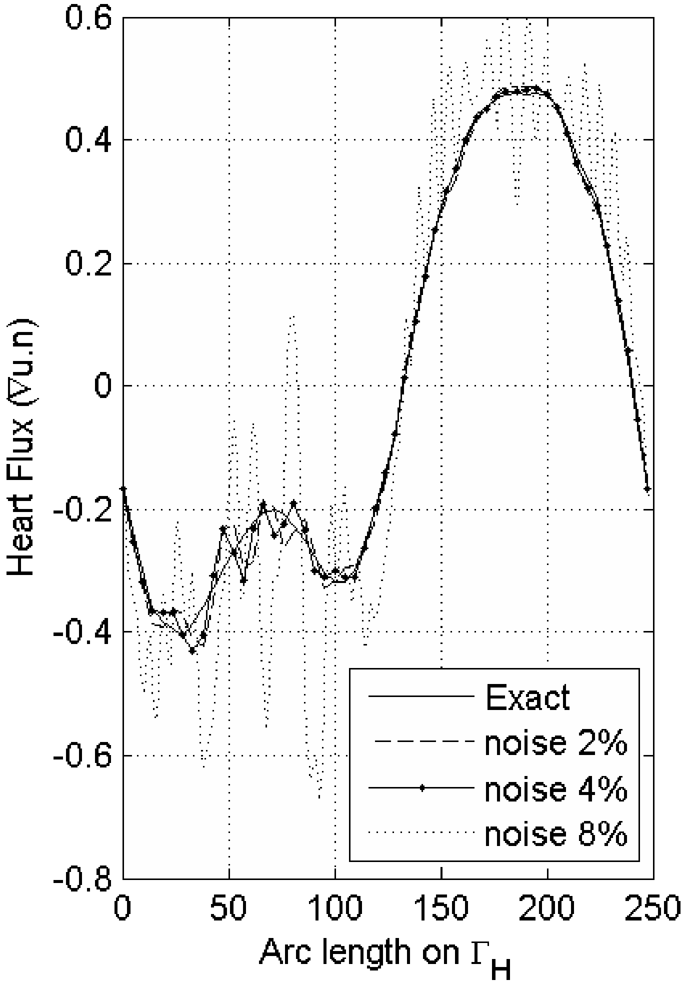
Simplified ECG



homogeneous isotropic case ($\sigma = 1$) and smooth boundary data

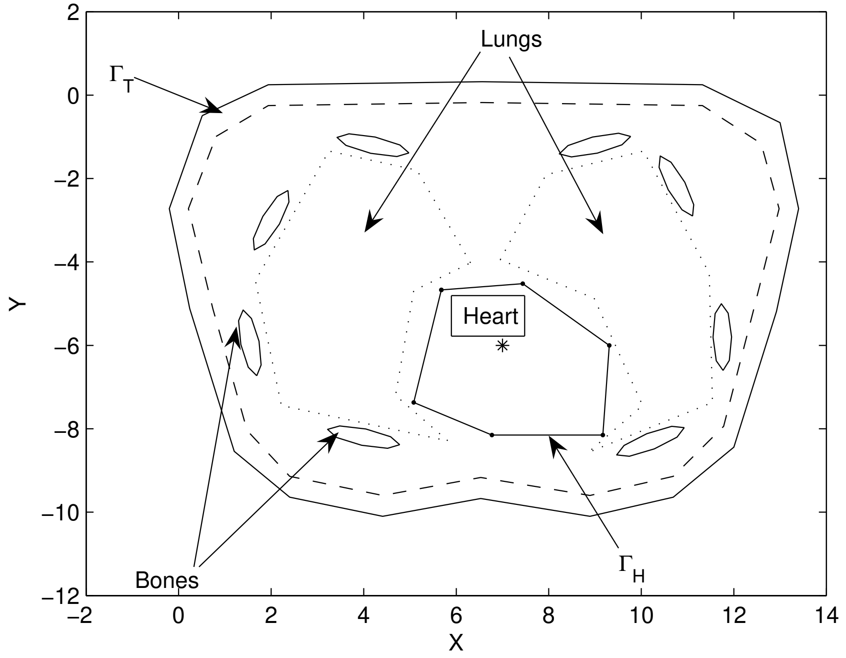
Numerical result for the smooth heart





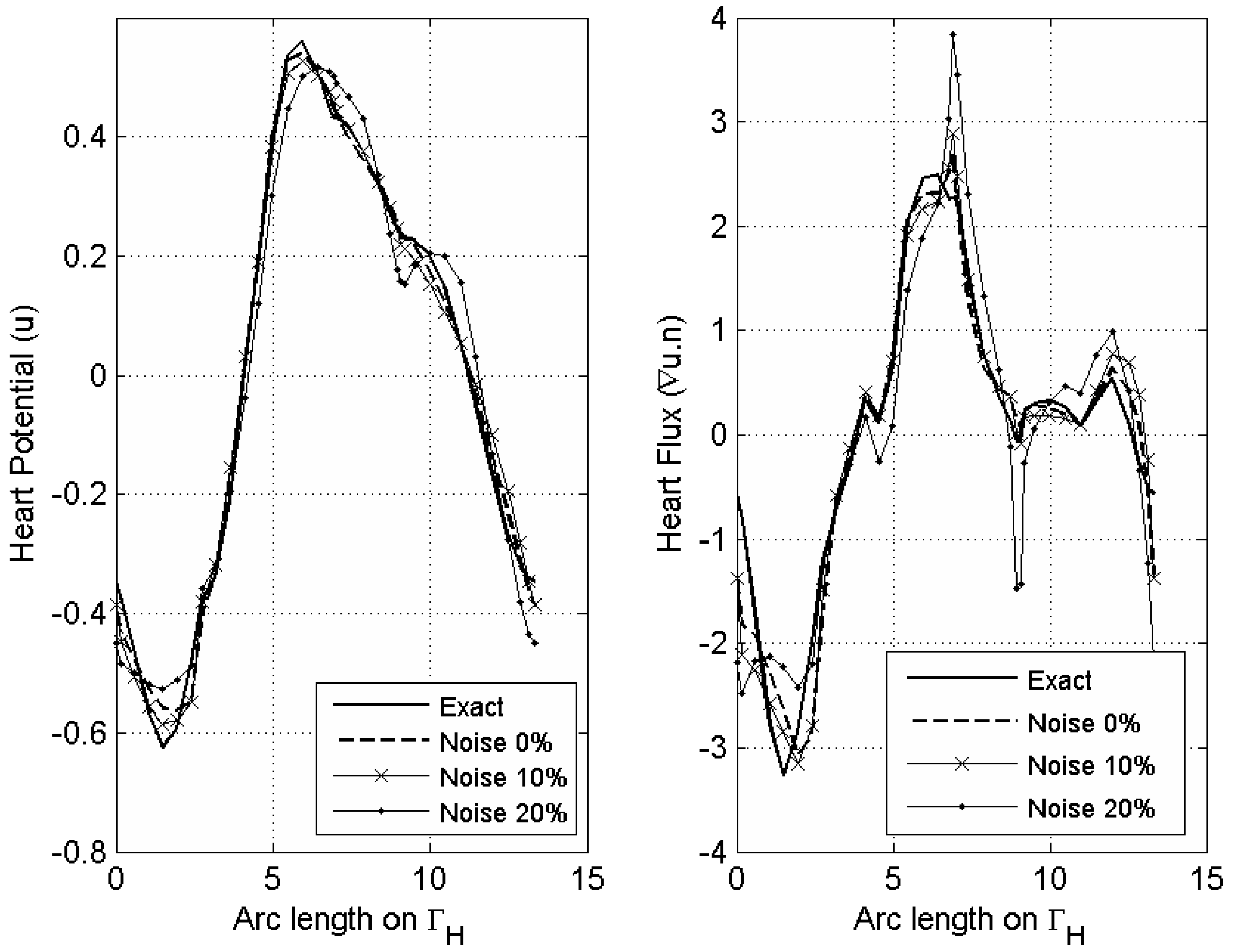
Numerical experiment with D-N and adjoint problems

Realistic ECG



heterogeneous anisotropic case and singular boundary data

Numerical result for the singular heart



Conclusion

Some features

- no need for regularization with energy formulation
- either interface equations or adjoint equations

Some perspectives

- comparison of approaches
- efficient solving algorithms