Nets Enriched over Closed Monoidal Structures

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Abstract. We show how the firing rule of Petri nets relies on a residuation operation for the commutative monoid of natural numbers. On that basis we introduce closed monoidal structures which are residuated monoids. We identify a class of closed monoidal structures (associated with a family of idempotent group dioids) for which one can mimic the token game of Petri nets to define the behaviour of these generalized Petri nets whose flow relations and place contents are valued in the closed monoidal structure.

1 Introduction

This paper reports on an ongoing research whose intent is to provide a uniform presentation of various families of Petri nets by recasting them as nets enriched over some algebraic structures, thus following the line of research best illustrated in [7] , a special issue of *Advances in Petri nets* dedicated to this theme. We aim at a general definition of nets parametric in algebraic structures corresponding to the kind of processes being modelled. We shall consider as a guideline the similar approaches that have been followed in the field of automata

theory. Moreover the similarities between the approach undertaken here and what have been considered with automata seems a prequisite to our long term objective of achieving an enriched theory of regions [3] following the categorical approach based on schizophrenic objects ([2], [1]).

In Section2 we compare two different algebraic approaches to generalized automata and justify the choice that we have made: a skeletal and non-commutative variant of Lawvere's generalized logics [8]. These structures that we term *Closed Monoidal Structures* in order to stress their connection to enriched category theory [9] are also known as *residuated monoids* [11] ; they are introduced in Section 3. In Section 4 we illustrate how Petri nets appear as nets enriched over the closed monoidal structure on the commutative monoid of integers. Based on this example we try to circumvent the class of properties that a closed monoidal structure should satisfy so that it could be associated with a family of Generalized Petri nets. This part of our work is reported in Section 5.

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2 Algebraic Approaches to Generalized Automata

The theory of Kleene algebras, by providing an axiomatization of regular expressions, has paved the way to an algebraic theory of automata. It gives a modelisation of the three fundamental operations of choice $(+)$, sequencing $(\bullet)$ and iteration $(\star)$ under the form of an idempotent semiring $(K, +, \bullet, 0, 1)$ together with specific axioms for iteration. When the semiring is complete, iteration can be obtained as a derived operation : $x^\star = \sum_{n \in N} x^n$. Any Kleene algebra gives rise to a special kind of automata. The crucial observation is that the set of square matrices of dimension n with entries in a Kleene algebra is also a Kleene algebra whose choice and sequencing operations are given respectively by $(M + N)_{i,j} = M_{i,j} + N_{i,j}$, and $(M \bullet N)_{i,j} = \sum_{1 \leq k \leq n} M_{i,k} \bullet N_{k,j}$. Iteration is somewhat more complicated to define however. It is then possible to define an automaton with n states

over a Kleene algebra K as a triple (λ, A, γ) where $\lambda \in K^{1 \times n}$ is the vector of initial states, $\gamma \in K^{n \times 1}$, the vector of final states, and $A \in K^{n \times n}$ is the transition matrix. This automaton then recognizes $\lambda \bullet A^\star \bullet \gamma \in K$, and the entry $A^\star(i,j) \in K$ can be interpreted as the "language" leading from state s_i to state s_j. For instance if we let $K = \wp(X^\star)$ be the set of languages over an alphabet X, with $A + B = A \cup B$, and $A \bullet B = \{uv \mid u \in A \;\&\; v \in B\}$ we obtain a complete Kleene algebra $(S, +, \bullet, 0, 1)$ with $0 = \emptyset$, the empty set, $1 = \{\varepsilon\}$, the language reduced to the empty word, and $A^\star = \cup_{n \in N} A^n$ the usual iteration on languages. Automata over this Kleene algebra are the usual finite automata. But one may also consider $K = \wp\left(X^2\right)$ with $A + B = A \cup B$, the union of relations, $A \bullet B = \{(x,y) \mid \exists z \in X \,.\, (x,y) \in A \;\&\; (y,z) \in B\}$, the composition of relations $0 = \emptyset$, the empty relation, and $1 = \Delta = \{(x,x) \mid x \in X\}$, the diagonal or identity relation. We then obtain another complete Kleene algebra with $A^\star$ the reflexive and transitive closure of relation A. Automata over this Kleene algebra can be interpreted as finite relational automata in which $(x,y) \in A^\star(i,j)$ if and only if there exists some path from state s_i to state s_j such that $(x,y) \in R$ where R is the relation obtained by composition of the various relations encountered along this path. Thus the relation recognized by the automaton can be interpreted as the input/output relation of the program whose control graph is given by the automaton. Many more (complete) Kleene algebras exist such for instance $(R_+ \cup \{\infty\}, \min, +, \infty, 1)$ for which $A^\star(i,j)$ gives the minimal cost of a path leading from state s_i to state s_j where the cost of a path is given by the sum of the values associated with each transition ; and $(R_+ \cup \{\infty\}, \max, \min, 0, \infty)$ for which $A^\star(i,j)$ gives the maximal flow of some path from state s_i to state s_j where the flow along some path is given by the minimum of the (maximal) flow enabled on each transition. Since $A^{\star\star} = A^\star$, the automaton $(\lambda, A^\star, \gamma)$ is equivalent to (i.e. recognizes the same element as) automaton (λ, A, γ). An automaton (λ, A, γ) such that $A^\star = A$ is said to be *saturated*.

A second approach to an algebraic description of generalized automata is to consider automata over an alphabet X and values in a semiring K, as introduced by Schützenberger [12] (see also [4]). We let $K\langle\langle X \rangle\rangle$ denote the set of formal power series with coefficients in K and set of variables X. Such a series is a map $s : X^\star \longrightarrow K$ that can be interpreted as a "generalized set" of words in which the degree of membership of a word is measured by an element of K (its coefficient). Indeed if K is the boolean semiring $K = \{0,1\}$ one has $K\langle\langle X\rangle\rangle = \wp(X^\star)$. The set of formal power series $K\langle\langle X\rangle\rangle$ is a semiring whose operations are given by : $(s+t)(w) = s(w) + t(w)$, $(s \bullet t)(w) = \sum_{w=uv} s(u) \bullet t(v)$. The same holds, by restriction, for its subset of polynomials $K\langle X\rangle = \{s \in K\langle\langle X\rangle\rangle \mid \exists^{<\infty} w \in X^\star\ s(w) \neq 0\}$(formal power series with a finite domain). Usually $K\langle\langle X\rangle\rangle$ is not a Kleene algebra, however a partially defined iteration operation exists. Indeed let say that a family of formal power series $\{s_i \in K\langle\langle X\rangle\rangle \mid i \in I\}$ is *locally finite* when $\forall w \in X^\star\ \exists^{<\infty} i \in I\ s_i(w) \neq 0$ then the sum of such a family can be defined : $(\sum s_i)(w) = \sum (s_i(w))$. For any proper

formal power series $(s(\varepsilon) = 0)$, the family $\{s^n \in K\langle\langle X\rangle\rangle \mid n \in N\}$ is locally finite, and if we let $s^\star = \sum_{n\in N} s^n$ then for any t the unique solution of the equation $x = t + sx$ (respectively of the equation $x = t + xs$) is $x = s^\star t$ (resp. $x = ts^\star$). Now an automaton over the semiring K consists of a finite alphabet X, an integer n (the dimension of the automaton), a morphism of monoids $\mu : X^\star \longrightarrow K^{n\times n}$, a vector $\lambda \in K^{1\times n}$ of initial states, and a vector $\gamma \in K^{n\times 1}$ of final states. This automaton recognizes the formal power series s such that $s(w) = \lambda \bullet \mu(w) \bullet \gamma$. The triple (λ, μ, γ) is called an n-dimensional linear representation of s, and s is said to be *recognizable.* Again, in the particular case where $K\langle\langle X\rangle\rangle = \wp(X^\star)$, it corresponds to the usual definitions known for finite automata. We say that a formal power series is *rational* if it belongs to the rational closure (i.e. closure by sum, product and star operation) of the semiring of polynomials. Then the theorem of Schützenberger stating that a formal power series is rational if and only if it is recognizable provides a generalization of Kleene theorem for finite automata. Probabilistic automata can also be associated similarly to the semiring $(R_+, +, \times, 0, 1)$, however extra conditions should be added constraining the definition of automata : an automaton consists of an initial distribution of probability λ (hence assumed to satisfy $\sum \lambda_i = 1$), $\mu_{i,j}(w)$ gives the probability to reach state s_j from state s_i when reading word $w \in X^\star$, and γ_i is the probability that state s_i be a final state, then if $M_{i,j} = \sum_{x\in X} \mu_{i,j}(x)$ represents the probability to reach state s_j from state s_i in one step, we further assume that (the normalisation property) : $\forall i\ \left(\sum_{1\leq j\leq n} M_{i,j} + \gamma_i = 1\right)$, i.e. either s_i is an accepting state or one can reach some other state in one step. Then the recognized formal power series is such that $p(w) = \lambda \bullet \mu(w) \bullet \gamma$ is the probability of recognizing word $w \in X^\star$.

A *dioid*[1] is a semiring for which the relation $x \leq y \Leftrightarrow \exists z\ : y = x + z$ is antisymmetric, i.e. is an order relation. This situation happens if the sum is idempotent, but also if the two following conditions are satisfied : the sum is cancellative $(x + y = x + z \Rightarrow y = z)$ and the neutral element cannot be decomposed as a sum $(x + y = 0 \Rightarrow x = y = 0)$.

[1] Some authors however use the term dioids for the restricted subclass of idempotent semirings.

Claim. In any dioid $x \vee y \leq x + y$, and a dioid is idempotent if and only if $x \vee y = x + y$

Proof. By definition of the order relation $x + y \geq x$ and $x + y \geq y$ and thus $x \vee y \leq x + y$. If $x \vee y = x + y$ then the sum is clearly idempotent. Conversely let us assume the idempotency of sum, observe that in that case the order relation can equivalently be defined as $x \leq y \Leftrightarrow y = x + y$. Indeed if $y = x + z$ then by idempotency : $y = x + x + z = x + y$ and the converse implication is immediate. Let z such that $x \leq z$ and $y \leq z$ then by the preceding remark $z = x + z = y + z$ and then $z = z + z = (x + z) + (y + z) = (x + y) + z$, i.e. $x + y \leq x \vee y$.

A dioid $(K, +, \times, 0, 1)$ is said to be complete if it is a complete lattice w.r.t. the induced order and for all $x \in K$ and $\{y_i \mid i \in I\} \subset K$, the following infinite distributive laws are satisfied :

$$x \bullet \left(\bigvee_{i \in I} y_i \right) = \bigvee_{i \in I} (x \bullet y_i) \quad \text{and} \quad \left(\bigvee_{i \in I} y_i \right) \bullet x = \bigvee_{i \in I} (y_i \bullet x)$$

A quantale (with unit) is a complete idempotent dioid. Since, by the above claim, the sum then coincides with the join, a quantale is usually presented as a structure $(K, \vee, \bullet, 1)$ consisting of a complete lattice with an infinitary joint operator $\vee$, and a monoid $(K, \bullet, 1)$ such that both infinite distributive laws are satisfied.

We saw that probabilistic automata are the automata over the semiring $(R_+, +, \times, 0, 1)$ satisfying certain normalisation properties. This semiring is a complete dioid but it is not idempotent and thus can certainly not be extended into a Kleene algebra structure. For a semiring to be part of a Kleene algebra it is necessary to be idempotent and sufficient to be idempotent and complete, i.e. a quantale. How do both notions of automata then compare ? It is readily verified that for any semiring K, the semiring $K \langle\langle X \rangle\rangle$ is idempotent (respectively is a quantale) whenever K is idempotent (respectively is a quantale). An automaton (λ, A, γ) over the quantale $K \langle\langle X \rangle\rangle$ viewed as a (complete) Kleene algebra is saturated if and only if $I + AA \leq A$ which reads as :

$$1 \leq A_{i,i}(\varepsilon) \quad \text{and} \quad \bigvee_{j} \bigvee_{uv=w} A_{i,j}(u) \bullet A_{j,k}(v) \leq A_{i,k}(w)$$

Since 1 is the greatest element the first condition reads as $A_{i,i}(\varepsilon) = 1$. Now matrix A is equivalent to the data $\mu : X^\star \longrightarrow K^{n \times n}$ where $\mu(w)(i, j) = A_{i,j}(w)$, and the preceding conditions rewrite as

$$\mu(\varepsilon)(i, i) = 1 \quad \text{and} \quad \bigvee_{uv=w} \bigvee_{j} \mu(u)(i, j) \bullet \mu(v)(j, k) \leq \mu(w)(i, k)$$

i.e. $\mu(\varepsilon) = I$ and $\mu(u) \bullet \mu(v) \leq \mu(w)$ whenever $uv = w$. Now $\mu : X^\star \longrightarrow K^{n \times n}$ is morphism of monoid, i.e. (λ, μ, γ) is an automaton over the semiring K with variables in X, if and only if

$$\bigvee_j A_{i,j}(u) \bullet A_{j,k}(v) = A_{i,k}(w) \quad \text{whenever} \quad uv = w$$

If K is the tropical semiring $(R \cup \{\infty\}, \min, +)$ the above identity says

$$\min_j (A_{i,j}(u) + A_{j,k}(v)) = A_{i,k}(w) \quad \text{whenever} \quad uv = w$$

i.e. the "distance" from state s_i to state s_k (associated with some word) is the minimal length of a path from s_i to s_k (labelled with the same word). In analogy with the corresponding notion borrowed from the theory of metric spaces, we thus term *geodesic* any saturated automaton (λ, A, γ) such that $\bigvee_j A_{i,j}(u) \bullet A_{j,k}(v) = A_{i,k}(w)$ whenever $uv = w$. We thus have established the following result :

Proposition 1. *If K is a quantale, a geodesic automaton (λ, A, γ) over the quantale $K\langle\langle X\rangle\rangle$ viewed as a (complete) Kleene algebra is the same thing as an automaton (λ, μ, γ) over the semiring K with variables in X, with the correspondence given by $\mu(w)(i,j) = A_{i,j}(w)$.*

Let us again consider the case of probabilistic automata. The semiring $K_1 = (R_+, +, \times, 0, 1)$ is a complete dioid hence it induces a quantale $Q = (R_+, \bigvee, \times, 1)$ however it should not be confused with that quantale. Probabilistic automata do constitute a class of automata over $Q\langle\langle X\rangle\rangle$ viewed as a (complete) Kleene algebra (because $x \vee y \leq x + y$) however, by forgetting the sum operation, we have lost all possibility of identifying this subclass of automata. In particular the class of geodesic automata over $Q\langle\langle X\rangle\rangle$ corresponds to the automata with variables X over the idempotent semiring $K_2 = (R_+, \vee, \times, 1)$ where $\vee$ is the least upper bound operation and these automata have no relation with probabilistic automata !

The above proposition compare the two different approaches in the particular case where they both apply (i.e. when K is a quantale). Since we would

like to be able to deal with the largest possible variety of generalized automata, we are rather searching for an algebraic structure that would allow to encompass these two approches. The closed monoidal structures introduced in the following section will realize this goal at least for large subclasses of dioids including all complete dioids and all group dioids.

3 Closed Monoidal Structures

A simple approach to generalized automata compatible with those described in the previous section is to identify such an automaton to a set of states S together with a map $A : S \times S \to K$ so that $A(s, s')$ measures the (possibly structured) set of trajectories from state s to state s' into some closed monoidal structure. This approach was first proposed by Lawvere [8] who termed them *generalized metric spaces.* As compared to this original definition, the approach taken here is in some respect more specific (because we use skeletal monoidal categories, here simply called monoidal structures) and in some other respect more general (because the tensor may not be symmetric in our case). An extension of Lawvere's approach to the non symmetric case was already proposed by Kasanghian, Kelly and Rossi (also in order to define generalized automata over closed monoidal categories). The presentation below can be seen as the simplification of their approach to the skeletical case.

Definition 1. *A* ***monoidal structure*** $\vartheta = (K, \leq, \otimes, 1)$ *is a set K equipped with an order relation $\leq$ and a structure of monoid $(K, \otimes, 1)$ where the composition $\otimes$, called tensor, is monotonic in both arguments. The structure is* ***closed*** *(or is a* ***residuated monoid****) when there exists two binary operators / and \\, called the right and left residual operations, verifying the universal property that for any elements x, y, and z of K one has*

$$x \leq z/y \;\Leftrightarrow\; x \otimes y \leq z \;\Leftrightarrow\; y \leq x \backslash z$$

Monotonicity of tensor follows from the existence of residuals (see [11]). The following identities are immediate consequences of the definition

$$\begin{array}{l}
x \leq x/1 \quad \text{and} \quad x \leq 1\backslash x \\
1 \leq x/x \quad \text{and} \quad 1 \leq x\backslash x \\
y/x \otimes x \leq y \quad \text{and} \quad x \otimes x\backslash y \leq y \\
(x' \leq x \quad \text{and} \quad y \leq y') \Rightarrow (y/x \leq y'/x' \quad \text{and} \quad x\backslash y \leq x'\backslash y') \\
z/\left(x \otimes y\right) = \left(z/y\right)/x \;\; \text{and} \quad \left(x \otimes y\right)\backslash z = y\backslash\left(x\backslash z\right) \\
x\backslash\left(z/y\right) = \left(x\backslash z\right)/y
\end{array}$$

When the tensor is commutative, the monoidal structure is termed *commutative*, we then usually adopt additive notations: $\oplus$ in place of $\otimes$, and 0 in place of 1. In that case both residual operations coincide and we denote it $\ominus$. By analogy to semigroups we say that a closed monoidal structure is *complete* if it is a complete lattice and the tensor is continuous in both arguments :

$$x \otimes \left(\bigvee_{i \in I} y_i\right) = \bigvee_{i \in I} (x \otimes y_i) \quad \text{and} \quad \left(\bigvee_{i \in I} y_i\right) \otimes x = \bigvee_{i \in I} (y_i \otimes x)$$

In that case the residuals are given by the formulas :

$$z/y = \bigvee \{x \mid x \otimes y \leq z\} \quad \text{and} \quad x\backslash z = \bigvee \{y \mid x \otimes y \leq z\}$$

hence there are derived operators and complete closed monoidal structures may be identified with continuous ordered monoids (this situation is similar to the identification of complete Kleene algebras with quantales). Let us enumerate some families of closed monoidal structures.

3.1 Complete Closed Monoidal Structures

As for Kleene algebras most closed monoidal structures will be complete. Nevertheless, as we shall see in the next sections, the residual operations are the basic operations needed for representing Petri net-like computations and, in order to remain as general as possible, we don't want to enforce completeness as long as this assumption is not strictly necessary. However let us mention some families of complete closed monoidal structures.

Complete Dioids. For instance, the complete dioid $(R_+ \cup \{\infty\}, +, \times, 0, 1)$ induces the commutative closed monoidal structure $(R_+ \cup \{\infty\}, \leq, \times, 1)$. whose residual is given by : $x \ominus y = x/y$ when $x, y \in R_+$, $\infty \ominus \infty = 0$, $\infty \ominus x = \infty$, and $x \ominus \infty = 0$, for any $x \in R_+$

Quantales. Quantales are just complete monoids whose sum is idempotent (equivalently coincides with join).

1. The most typical example (already mentioned) is the set $K = \wp(X^\star)$ of languages on an alphabet X with set-theoretic inclusion as order relation, concatenation of languages as tensor product

$$L \otimes M = \{u.v \in X^* \mid u \in L \ \wedge \ v \in M\}$$

$1 = \{\varepsilon\}$, the language reduced to the empty word, as unit, and the usual residual operations on languages :

$$N/L = \{v \in X^* \mid \forall u \in L \ \ v.u \in N\} \text{ and } L \backslash N = \{v \in X^* \mid \forall u \in L \ \ u.v \in N\}$$

This monoidal structure stems from the quantale $(\wp(X^\star), \cup, \otimes, \emptyset, \{\varepsilon\})$. Similarly one can obtain a closed monoidal structure $\wp(\mathcal{M})$ by replacing the free monoid $X^\star$ by an arbitrary monoid $\mathcal{M}$.

2. Similarly the set $K = rel(U)$ of binary relations over a set U (for universe) is also a closed monoidal structure with set-theoretic inclusion as order relation, composition of relations as tensor product

$$R \otimes S = \{(x, z) \in U \times U \mid \exists x \in U \quad (x, y) \in R \ \wedge \ (y, z) \in S\}$$

the diagonal $\Delta = \{(x, x) \in U \times U \mid x \in U\}$ as unit, and the residuals operations on relations :

$$R/S = \{(x, y) \in U \times U \mid \forall z \in U \quad (y, z) \in S \ \Rightarrow \ (x, z) \in R\}$$

$$S \backslash R = \{(x, y) \in U \times U \mid \forall z \in U \quad (z, x) \in S \ \Rightarrow \ (z, y) \in R\}$$

This monoidal structure stems from the quantale $(rel(U), \cup, \otimes, \emptyset, \Delta)$. The set of binary relations is also equipped with two unary operators giving respectively the inverse $R^{-1} = \{(x, y) \in R \mid \ (y, x) \in R\}$ and the complement $R^c = \{(x, y) \in U \times U \mid \ (y, x) \notin R\}$ of a relation R. One may then check that residuals can be expressed using these operators and composition as :

$R/S = \left(R^c \otimes S^{-1}\right)^c$ and $S\backslash R = \left(S^{-1} \otimes R^c\right)^c$. The calculus of binary relations has a long history involving works of famous logicians like De Morgan, Russel and Tarski (see the survey by Pratt [10]), and there is a recent increasing interest in computer science community due to its similarities with the Lambek Calculus (a logical calculus used for the treatment of natural languages) and linear logic [11].

3. Our third example corresponds to Petri nets: the set of non negative integers with addition, and the opposite of the usual order relation is a commutative closed monoidal structure whose residuals are given by truncated difference : $x \ominus y = x - y$ if $x \geq y$ and $x \ominus y = 0$ otherwise:

$$x + y \geq z \;\Leftrightarrow\; y \geq z \ominus x$$

If we add an element ∞ such that $x \leq \infty$, and $\infty \ominus x = \infty$ for every $x \in N$, and $x + \infty = \infty + x = \infty$, and $x \ominus \infty = 0$ for every $x \in N \cup \{\infty\}$ we obtain a complete commutative closed monoidal structure which stems from the quantale $(N \cup \{\infty\}, \min, +, \infty, 0)$.

Complete Heyting Algebras. A complete algebra is a quantale whose tensor coincides with the meet of the lattice. It then induces a commutative closed monoidal structure whose residual $y \ominus x$ is the relative complement $x \Rightarrow y = \bigvee \{z \mid x \wedge z \leq y\}$. The universal property of residuals: $x \wedge y \leq z \Leftrightarrow y \leq x \Rightarrow z$ is modus ponens which explain why Lawvere coined the term *generalized logics.* A boolean ring $\Re$ of sets ordered by inverse inclusion, with set union as tensor is a commutative closed monoidal structure whose residuals are given by set-theoretic difference :

$$X \cup Y \supseteq Z \;\Leftrightarrow\; Y \supseteq Z \backslash X$$

This monoidal structure stems from the complete Heyting algebra $(\Re, \supseteq)$.

3.2 Group-Like Monoidal Structures

In many respects closed monoidal structures appear as weak forms of groups, namely any group $\left(G, \otimes, 1, (.)^{-1}\right)$ is a closed monoidal structure whose order relation is equality and whose residuals are given by : $x/y = x \otimes y^{-1}$ and $y\backslash x = y^{-1} \otimes x$. A group $\left(G, \bullet, 1, (.)^{-1}\right)$ is said to be partially ordered if there exists a partial order $\leq$ with respect to which the product is monotonic in both arguments. A non trivial group can be ordered by letting $x < y \Leftrightarrow x \in y \bullet N$ if and only there exists a non-empty subset $N \subset G$ such that $N \cap N^{-1} = \emptyset$, $N \bullet N \subseteq N$, and $x \bullet N = N \bullet x$ for all $x \in G$ and in that case $N = \{x \in G \mid x < 1\}$. We then obtained a closed monoidal structure $(K, \leq, \otimes, 1)$, where $K = N \cup \{\bot, \top\}$ where $\bot$ and $\top$ new elements (i.e. not elements of G). The order relation is given by $x \leq y \Leftrightarrow x < y \;\vee\; x = y$ for $x, y \in N$ (order induced from the order on G) and $\bot \leq x \leq \top$ for all $x \in N$. The tensor is given by $x \otimes y = x \bullet y$ if

$x, y \in N$, $x \otimes \bot = \bot \otimes x = \bot$, and $x \otimes \top = \top \otimes x = x$ for all $x \in K$ (i.e. the least element is absorbing and the greatest element is neutral). The residuals are given as follows where a, b range over N, x ranges over K , and y ranges over $K \diagdown \{\bot\}$: $a/b = a \bullet b^{-1}$ and $b \backslash a = b^{-1} \bullet a$ if $a \leq b$ and else $a/b = b \backslash a = \top$; $x/\top = \top \backslash x = x$; $x/\bot = \bot \backslash x = \top$; $\top/x = x \backslash \top = \top$; $\bot/y = y \backslash \bot = \bot$. The details of the verification are easy and left to the reader. It can also be verified that with the prescribed order on K, the above is the unique possible closed monoidal structure on K such that for all a, b in N :

$$a/b = a \bullet b^{-1} \quad \text{and} \quad b \backslash a = b^{-1} \bullet a \quad \text{if} \quad a \leq b \quad \text{else} \quad a/b = b \backslash a = \top$$

Notice that the element $1 \in G$ could have play the role of $\top$ since it satisfies the required properties w.r.t. to the elements in N : it is a neutral element for $\otimes$, and a greatest element for the order. The group of integers $(Z, +, 0)$ with its usual ordering induces in this way a closed monoidal structure isomorphic to the one associated with the quantale $(N \cup \{\infty\}, \min, +, \infty, 0)$. This suggests to look at the particular case where the order is induced from a sum.

A dioid $(K, \oplus, \otimes, \varepsilon, e)$ is a *group dioid* if every element of $K \setminus \{\varepsilon\}$ has an inverse for $\otimes$. Hence $(K \setminus \{\varepsilon\}, \otimes, e)$ is a group ordered by the relation induced by the sum : $x \leq y \Leftrightarrow \exists z \ : y = x \oplus z$. Now since ε is neutral for $\oplus$, it is a least element for this order and we can verify that it is an absorbing element for the tensor. Hence it can play the role of $\bot$ in the above construction, i.e. one has

Proposition 2. *Any group dioid* $(K, \oplus, \otimes, \varepsilon, e)$ *induces a closed monoidal structure* $(M, \leq, \otimes, e)$ *where* M *is the interval* $]\varepsilon; e] = \{x \in K \mid \varepsilon < x \leq e\}$ *and* $\leq$ *and* $\otimes$ *are respectively the order and the tensor of the dioid induced on this interval. The residuals are given by* $a/b = a \otimes b^{-1}$ *and* $b \backslash a = b^{-1} \otimes a$ *if* $a \leq b$ *and else* $a/b = b \backslash a = e$ *; and* $x/e = e \backslash x = x$*, and* $e/x = x \backslash e = e$ *for any* $a, b \in]\varepsilon; e[$*, and* $x \in]\varepsilon; e]$*. If the dioid is idempotent (sum is join :* $x \oplus y = x \vee y$*), then it is a lattice whose meet is given by* $x \wedge y = \left(x^{-1} \vee y^{-1}\right)^{-1}$ *and then residuations are given by :* $a/b = e \wedge \left(a \otimes b^{-1}\right)$ *and* $b \backslash a = e \wedge \left(b^{-1} \otimes a\right)$*.*

It induces also a *complete* closed monoidal structure $(M', \leq, \otimes, e)$ where M' is the interval $[\varepsilon; e] = \{x \in K \mid \varepsilon \leq x \leq e\}$. The residuals further satisfy : $x/\varepsilon = \varepsilon \backslash x = e$ and $\varepsilon/y = y \backslash \varepsilon = \varepsilon$ for $x \in [\varepsilon; e]$, and $y \in]\varepsilon; e]$.

4 Nets over a Closed Monoidal Structure

Petri nets are associated with the monoidal structure $(N, \leq, +, 0)$ where $\leq$ is the opposite of the usual order $\sqsubseteq$ on N. As we saw it is a commutative closed monoidal structure whose residual is given by the truncated difference : $x \ominus y = x - y$ if $x \geq y$ else $x \ominus y = 0$. Indeed a Petri net is a structure $(P, E, Pre, Post)$ where P is a finite set of places, E a finite set of events (disjoint from P), and $Pre, Post : E \to N^P$ are called the flow relations. We can inductively extend these maps : $Pre, Post : E^* \to N^P$ by letting $Pre(\varepsilon)(p) = Post(\varepsilon)(p) = 0$ and

$$\begin{aligned} Pre(ue) &= [Pre(e) \ominus Post(u)] + Pre(u) \\ Post(ue) &= [Post(u) \ominus Pre(e)] + Post(a) \end{aligned}$$

with a componentwise definition of the closed monoidal strcuture on N^P. A marking is a map from P into N. We readily verify that

$$M\,[u\rangle\,M' \iff M \sqsupseteq Pre(u) \quad \wedge \; M' = [M \ominus Pre(u)] + Post(u)$$

where $M\,[u\rangle\,M'$ is the usual firing relation for Petri nets and where all operators are defined componentwise. Moreover reversibilty can be expressed by the fact that

$$M\,[u\rangle\,M' \iff M' \sqsupseteq Post(u) \quad \wedge \; M = [M' \ominus Post(u)] + Pre(u)$$

Another equivalent formulation is

$$M\,[u\rangle\,M' \iff M \sqsupseteq Pre(u) \quad \wedge \; M' \sqsupseteq Post(u) \quad \wedge \; M \ominus Pre(u) = M' \ominus Post(u)$$

As in the above equivalences, when speaking of the firing rule of nets we shall allow ourselves to use the *notation* $\sqsupseteq$ in place of $\leq$, the order relation of the closed monoidal structure, since the order relation $\sqsubseteq = (\leq)^{-1}$ better reflects the intuition when dealing with nets. One can argue that we could have use this relation in the first place by using the equivalences

$$x \sqsupseteq z/y \;\Leftrightarrow\; x \otimes y \sqsupseteq z \;\Leftrightarrow\; y \sqsupseteq x\backslash z$$

for the definition of closed monoidal structures. We have indeed hesitated between these two possibilities for a long time and have finally taken the choice that allows a better comparison with semirings and dioids. This reversing of order relations reflects the duality between automata and nets that we intend to investigate in the future.

A sequence $u \in E^*$ of events can always be fired in some marking : indeed it can be fired in any marking M such that $M \sqsupseteq Pre(u)$. We assume $Pre(u)$ to give the "minimal amount of resources" in places so that the sequence $u \in E^*$ is firable. For some classes of nets however, like Elementary Net Systems, there exists sequences of events that are firable in no markings. A value $x \sqsupseteq Pre(u)$ where u is such an unfirable sequence should not be a legitimate place value. We can indeed handle Elementary Net Systems in this way. But making a distinction between the set of place contents and the set of flow relation values raises new issues that we can treat in this particular case and in similar cases, but for which we have not yet a global satisfactory answer. We propose then in the present paper to stick to base case where place contents and flow relation values belongs to the same set.

Definition 2. *A net over a closed monoidal structure $\vartheta = (K, \sqsupseteq, \otimes, 1)$ is a structure $(P, E, Pre, Post)$ where P is a finite set of places, E a finite set of events (disjoint from P), and $Pre, Post : E \to K^P$ are called the flow relations.*

A marking is a map $M : P \to K$. An event $e \in E$ is said to be firable in marking M and leads then to marking M', in notation $M\,[e\rangle\,M'$, if

$$M \sqsupseteq Pre(e) \quad \wedge \; M' = M \diagup Pre(e) \otimes Post(e)$$

however in order to meet intuition it should be the case that the first condition of this conjunct states that $e \in E$ is firable in marking M, and the second part gives the computation of the resulting marking M'. The interpretation is that when $M \sqsupseteq Pre(e)$, then marking M should decompose as $M = M \diagup Pre(e) \otimes Pre(e)$; hence assuming

$$a \sqsupseteq b \Rightarrow a = (a/b) \otimes b$$

In order to get marking M', we then replace $Pre(e)$ by $Post(e)$ in the above decomposition of marking M. Due to the curryfying law of closed monoidal structures (namely $z/\left(x \otimes y\right) = \left(z/y\right)/x$) one can interpret z/u as "popping u from z" ; hence marking M' is obtained by popping $Pre(e)$ from marking M and then pushing $Post(e)$: $z \otimes (x \otimes y) = (z \otimes x) \otimes y$. In order to keep reversibility we need to ensure that both operations of popping and pushing are invertible, which amounts to the condition : $(a \otimes b) \diagup b = a$. Then the firing rule can equivalently be stated as

$$M\,[e\rangle\,M' \iff M' \sqsupseteq Post(e) \quad \wedge \; M = M' \diagup Post(e) \otimes Pre(e)$$

by reversing the roles of $Post(e)$ and $Pre(e)$; or still as

$$M\,[e\rangle\,M' \iff M \sqsupseteq Pre(e) \quad \wedge M' \sqsupseteq Post(e) \quad \wedge \; M \diagup Pre(e) = M' \diagup Post(e)$$

Of course we could also chose a fifo structure (first in first out) rather than a lifo (last in first out) structure for places in which case we need the following equivalences to hold true :

$$\begin{aligned} M\,[e\rangle\,M' &\iff M \sqsupseteq Pre(e) \quad \wedge \; M' = Post(e) \otimes M \diagup Pre(e) \\ &\iff M' \sqsupseteq Post(e) \quad \wedge \; M = Post(e) \diagdown M' \otimes Pre(e) \\ &\iff M \sqsupseteq Pre(e) \quad \wedge \; M' \sqsupseteq Post(e) \quad \wedge \; M \diagup Pre(e) = Post(e) \diagdown M' \end{aligned}$$

this further requires:

$$\begin{aligned} &a \sqsupseteq b \Rightarrow a = b \otimes (b \backslash a) \\ &b \diagdown (b \otimes a) = a \end{aligned}$$

Now comes the problem of extending inductively the definitions of $Pre(u)$ and $Post(u)$ for words $u \in E^*$ so that

$$\begin{aligned} M\,[u\rangle\,M' &\iff M \sqsupseteq Pre(u) \quad \wedge \; M' = M \diagup Pre(u) \otimes Post(u) \\ &\iff M' \sqsupseteq Post(u) \quad \wedge \; M = M \diagup Post\,(u) \otimes Pre(u) \\ &\iff M \sqsupseteq Pre(u) \quad \wedge \; M' \sqsupseteq Post(u) \quad \wedge \; M \diagup Pre(u) = M \diagup Post\,(u) \end{aligned}$$

(and similarly for the fifo case) where $M \sqsupseteq Pre(u)$ is equivalent to $M\,[u\rangle$ (the enabling of sequence u in marking $M : \exists M' \;\; M\,[u\rangle\, M'$) and $M' \sqsupseteq Post(u)$ is dually equivalent to $[u\rangle M'$ (the co-enabling of sequence u in marking $M' : \exists M \;\; M\,[u\rangle\, M'$). For that purpose we proceed to some computations :

$$\begin{aligned} M\,[uv\rangle &\iff M \sqsupseteq Pre(u) \quad \wedge \; M \diagup Pre(u) \otimes Post(u) \sqsupseteq Pre(v) \\ &\iff M \sqsupseteq Pre(u) \quad \wedge \; M \diagup Pre(u) \sqsupseteq Pre(v) \diagup Post(u) \\ &\implies M \sqsupseteq (Pre(v) \diagup Post(u)) \otimes Pre(u) \end{aligned}$$

The first equivalence is how we would like $M\,[uv\rangle$ to be defined (this will prove to be independent of the decomposition of $w = uv$ by associativity of the binary relation to be defined below). The second equivalence is just an application of residuation, and the last implication comes from the equivalence

$$M \sqsupseteq Pre(u) \iff M = M \diagup Pre(u) \otimes Pre(u)$$

This suggests the definition $Pre(uv) = (Pre(v) \diagup Post(u)) \otimes Pre(u)$ and similarly $Post(uv) = (Post(u) \diagup Pre(v)) \otimes Post(v)$. Now this requires that the above implication is an equivalence, i.e.

$$a \sqsupseteq b \otimes c \iff [a \sqsupseteq c \;\wedge\; a/c \sqsupseteq b]$$

for every a, b, and c in K. In the fifo case we should also add the requirement that

$$a \sqsupseteq b \otimes c \iff [a \sqsupseteq b \;\wedge\; b\backslash a \sqsupseteq c]$$

and let $Pre(uv) = (Post(u) \diagdown Pre(v)) \otimes Pre(u)$ and $Post(uv) = Post(v) \otimes (Post(u) \diagup Pre(v))$ since in that case :

$$\begin{aligned} M\,[uv\rangle &\iff M \sqsupseteq Pre(u) \quad \wedge \; Post(u) \otimes M \diagup Pre(u) \sqsupseteq Pre(v) \\ &\iff M \sqsupseteq Pre(u) \quad \wedge \; M \diagup Pre(u) \sqsupseteq Post(u) \diagdown Pre(v) \\ &\iff M \sqsupseteq (Post(u) \diagdown Pre(v)) \otimes Pre(u) \end{aligned}$$

and

$$\begin{aligned} [uv\rangle\, M' &\iff M' \sqsupseteq Post(v) \quad \wedge \; Post(v) \diagdown M' \otimes Pre(v) \sqsupseteq Post(u) \\ &\iff M' \sqsupseteq Post(v) \quad \wedge \; Post(v) \diagdown M' \sqsupseteq Post(u) \diagup Pre(v) \\ &\iff M' \sqsupseteq Post(v) \otimes (Post(u) \diagup Pre(v)) \end{aligned}$$

We can then define a binary operation on $K \times K$ as follows. We denote Pre and $Post$ the two canonical projections from $K \times K$ to K, so that an element $\alpha \in K \times K$ be written $\alpha = (Pre(\alpha), Post(\alpha))$ and the binary relation $\boxtimes$ is defined by the two above identities (for the lifo case)

$$\begin{aligned} Pre(\alpha \boxtimes \beta) &= (Pre(\beta) \diagup Post(\alpha)) \otimes Pre(\alpha) \\ Post(\alpha \boxtimes \beta) &= (Post(\alpha) \diagup Pre(\beta)) \otimes Post(\beta) \end{aligned}$$

K should contain a least element $\perp$ (for $\sqsubseteq$) corresponding to no constraint ($M(p) \sqsupseteq \perp$ is always satisfied) ; and $(\perp, \perp)$ should be the neutral element of this composition, so that we can let $Pre(\varepsilon)(p) = Post(\varepsilon)(p) = \perp$. Then we can inductively define the maps $Pre, Post : E^* \to K^P$ by letting $(Pre(u)(p), Post(u)(p)) = \varphi(u)(p)$ where $\varphi : E^* \to \left(K^2\right)^P$ is the unique morphism of monoids such that the images $\varphi(e)(-) = (Pre(e)(-), Post(e)(-))$ of the generators $e \in E$ are given by the flow relations of the net. We then obtain

$$
\begin{aligned}
&Pre(\varepsilon)(p) = Post(\varepsilon)(p) = \perp \\
&Pre(u \otimes v)(p) = (Pre(v)(p) \,/\, Post(u)(p)) \otimes Pre(u)(p) \\
&Post(u \otimes v)(p) = (Post(u)(p) \,/\, Pre(v)(p)) \otimes Post(v)(p)
\end{aligned}
$$

5 Petri Monoidal Structures

The following definition provides a set of conditions sufficient to ensure the different requirements mentioned in the previous section. We omit most of the proofs of the various following propositions. Hint of most of these proofs and their main arguments were already sketched in the informal discussion of the previous section. An important point however that was not touched upon in that discussion and that we shall consider here is how associativity of operation $\boxtimes$ is ensured.

Definition 3. *A Petri monoidal structure is a closed monoidal structure* $\vartheta = (K, \sqsupseteq, \otimes, 1)$ *such that :*

1. 1 *is the least element of* $(K, \sqsubseteq)$
2. $(a \sqsupseteq b \otimes c) \Longleftrightarrow (a \sqsupseteq c \ \wedge \ a/c \sqsupseteq b) \Longleftrightarrow (a \sqsupseteq b \ \wedge \ b\backslash a \sqsupseteq c)$
3. $(a \otimes b)/b = a$ *and* $b\backslash(b \otimes a) = a$

Proposition 3. *A Petri monoidal structure satisfies the following:*

1. *There are no divisor of the unit :* $[a \otimes b = 1 \Rightarrow a = b = 1]$, *and* $1/a = a\backslash 1 = 1$; $a/1 = 1\backslash a = a$; $a/a = a\backslash a = 1$; *and* $a \sqsupseteq b \Leftrightarrow b/a = 1 \Leftrightarrow a\backslash b = 1$.
2. *It has a join given by* $a \vee b = a/b \otimes b = b \otimes b\backslash a = b/a \otimes a = a \otimes a\backslash b$.
3. *It has a meet given by* $a \wedge b = a/(b\backslash a) = (a/b)\backslash a = b/(a\backslash b) = (b/a)\backslash b$.
4. $a \sqsupseteq b \Longrightarrow (a = a/b \otimes b = b \otimes b\backslash a)$.
5. $a \otimes b \sqsupseteq a$; $a \otimes b \sqsupseteq b$.

Definition 4. *A Petri monoidal structure is said to be* lifo *when*

$$(b \otimes c)/a = b/(a/c) \otimes c/a \quad \text{and} \quad a\backslash(b \otimes c) = a\backslash b \otimes (b\backslash a)\backslash c$$

It is said to be fifo *when :*

$$(b \otimes c)/a = b/a \otimes c/(b\backslash a) \quad \text{and} \quad a\backslash(b \otimes c) = (a/c)\backslash b \otimes a\backslash c$$

If the tensor is commutative the lifo and fifo conditions are equivalent and can be expressed (with the additive notation) as :

$$(b \oplus c) \ominus a = [b \ominus (a \ominus c)] \oplus (c \ominus a)$$

but of course the terminology "lifo" and "fifo" makes no much sense in that case.

The respective conditions in the above definition are both "internalisations" of condition (2) in Def.3, this fact is expressed by the following two propositions.

Proposition 4. *A lifo Petri monoidal structure is a closed monoidal structure* $\vartheta = (K, \sqsupseteq, \otimes, 1)$ *such that :*

1. 1 *is the least element of* $(K, \sqsupseteq)$.
2. *There are no divisor of the unit :* $[a \otimes b = 1 \Rightarrow a = b = 1]$.
3. $(a \otimes b)/b = a$ *and* $b \backslash (b \otimes a) = a$.
4. $(b \otimes c)/a = b/(a/c) \otimes c/a$ *and* $a \backslash (b \otimes c) = a \backslash b \otimes (b \backslash a) \backslash c$

Proof. By the residuation property $(a \sqsupseteq b \otimes c) \Longleftrightarrow [(b \otimes c)/a = 1] \Longleftrightarrow [b/(a/c) \otimes c/a = 1]$. Since $[a \otimes b = 1 \Rightarrow a = b = 1]$, this latter condition is equivalent to $[b/(a/c) = 1] \wedge [c/a = 1]$; i.e. $(a \sqsupseteq c \ \wedge \ a/c \sqsupseteq b)$.
The equivalence $(a \sqsupseteq b \otimes c) \Longleftrightarrow (a \sqsupseteq b \ \wedge \ b \backslash a \sqsupseteq c)$ is proved similarly

In the same manner we obtain the analogous proposition :

Proposition 5. *A fifo Petri monoidal structure is a closed monoidal structure* $\vartheta = (K, \sqsupseteq, \otimes, 1)$ *such that :*

1. 1 *is the least element of* $(K, \sqsupseteq)$.
2. *There are no divisor of the unit :* $[a \otimes b = 1 \Rightarrow a = b = 1]$.
3. $(a \otimes b)/b = a$ *and* $b \backslash (b \otimes a) = a$.
4. $(b \otimes c)/a = b/a \otimes c/(b \backslash a)$ *and* $a \backslash (b \otimes c) = (a/c) \backslash b \otimes a \backslash c$.

Proposition 6. *A Petri monoidal structure* $\vartheta = (K, \sqsupseteq, \otimes, 1)$ *with a total ordering is lifo.*

Corollary 1. *The closed monoidal structure induced by an idempotent group dioid with a total ordering is a lifo Petri monoidal structure.*

This corollary provides a reasonable class of group-like closed monoidal structures that are lifo Petri monoidal structures. Unfortunately we don't have an analogue of Proposition6 (and hence of its corollary) in the fifo case. Condition 4 in Prop. 5 is much harder to establish. And this is easy to understand : as long as we pop and push "on the same side", things go quite nicely with group-like closed monoidal structures (in which the right residuals looks like ab^{-1} with the negative part "consuming the data" coming first into the lifo due to the curryfication law : $u/\left(ab^{-1}\right) = \left(u/b^{-1}\right)/a$. When on the contrary consumption and production of data are made at the opposite sides of a fifo, we need to express how a data just entered can have an effect on the enabling or on the contrary on the inhibition of some event. This information need to "flow" through the entire fifo and take into account some parts of the value of the place that comes from its past history. This flow of information is obtained by mixing both residual operations as described by the Condition 4 in Prop. 5. And it is very hard to find concrete models that realize this computation.

Proposition 7. *Let $\vartheta = (K, \sqsupseteq, \otimes, 1)$ be a fifo Petri monoidal structure . We have a monoid $\left(K^2, \boxtimes, e\right)$ with unit $e = (1,1)$ and whose composition law is given by :*

$$(u, u') \boxtimes (v, v') = (v/u' \otimes u, u'/v \otimes v')$$

Proof. It is immediate that $e = (1,1)$ is a left end right unit of the given product. Let us prove the associativity of the operation.

$$\begin{aligned} [(u, u') \boxtimes (v, v')] \boxtimes (w, w') &= (v/u' \otimes u, u'/v \otimes v') \boxtimes (w, w') \\ &= (w/\left[u'/v \otimes v'\right] \otimes v/u' \otimes u, \left[u'/v \otimes v'\right]/w \otimes w') \end{aligned}$$

$$\begin{aligned} (u, u') \boxtimes [(v, v') \boxtimes (w, w')] &= (u, u') \boxtimes (w/v' \otimes v, v'/w \otimes w') \\ &= (\left[w/v' \otimes v\right]/u' \otimes u, u'/\left[w/v' \otimes v\right] \otimes v'/w \otimes w') \end{aligned}$$

For the left-hand side:

$$\begin{aligned} \left[w/v' \otimes v\right]/u' \otimes u =& (w/v')/(u'/v) \otimes v/u' \otimes u :: (b \otimes c)/a = b/(a/c) \otimes c/a \\ =& w/\left[u'/v \otimes v'\right] \otimes v/u' \otimes u \quad :: (a/b)/c = a/(c \otimes b) \end{aligned}$$

For the right-hand side:

$$\begin{aligned} \left[u'/v \otimes v'\right]/w \otimes w' =& (u'/v)/(w/v') \otimes v'/w \otimes w' :: (b \otimes c)/a = b/(a/c) \otimes c/a \\ =& u'/\left[w/v' \otimes v\right] \otimes v'/w \otimes w' \quad :: (a/b)/c = a/(c \otimes b) \end{aligned}$$

We have the same result in case of the fifo semantics. In that case the composition is given by

$$(u, u') \boxtimes (v, v') = (u' \backslash v \otimes u, v' \otimes u'/v)$$

and then

$$\begin{aligned} [(u, u') \boxtimes (v, v')] \boxtimes (w, w') &= (u' \backslash v \otimes u, v' \otimes u'/v) \boxtimes (w, w') \\ &= (\left[v' \otimes u'/v\right] \backslash w \otimes u' \backslash v \otimes u, w' \otimes \left[v' \otimes u'/v\right]/w) \end{aligned}$$

$$\begin{aligned}(u,u')\boxtimes[(v,v')\boxtimes(w,w')] &= (u,u')\boxtimes(v'\backslash w\otimes v, w'\otimes v'/w)\\ &= (u'\backslash[v'\backslash w\otimes v]\otimes u, w'\otimes v'/w\otimes u'/[v'\backslash w\otimes v])\end{aligned}$$

For the left-hand side:

$$\begin{aligned}u'\backslash[v'\backslash w\otimes v]\otimes u =& (u'/v)\backslash(v'\backslash w)\otimes u'\backslash v\otimes u :: a\backslash(b\otimes c)=(a/c)\backslash b\otimes a\backslash c\\ =& [v'\otimes u'/v]\backslash w\otimes u'\backslash v\otimes u \;\; :: (c\otimes b)\backslash a = b\backslash(c\backslash a)\end{aligned}$$

For the right-hand side:

$$\begin{aligned}w'\otimes[v'\otimes u'/v]/w =& w'\otimes v'/w\otimes(u'/v)/(v'\backslash w) :: (b\otimes c)/a = b/a\otimes c/(b\backslash a)\\ =& w'\otimes v'/w\otimes u'/[v'\backslash w\otimes v] \;\; :: a/(b\otimes c)=(a/c)/b\end{aligned}$$

Let $(P,E,Pre,Post)$ be a net over a lifo Petri monoidal structure $\vartheta = (K,\sqsupseteq,\otimes,1)$. We then can define $Pre,Post : E^* \to K^P$ by letting $(Pre(u)(p), Post(u)(p)) = \varphi(u)(p)$ where $\varphi : E^* \to \left(K^2\right)^P$ is the unique morphism of monoids such that the images $\varphi(e)(-) = (Pre(e)(-), Post(e)(-))$ of the generators $e \in E$ are given by the flow relations of the net. We then obtain

$$\begin{aligned}&Pre(\varepsilon)(p) = Post(\varepsilon)(p) = 1\\ &Pre(u\otimes v)(p) = (Pre(v)(p) \diagup Post(u)(p))\otimes Pre(u)(p)\\ &Post(u\otimes v)(p) = (Post(u)(p)\diagup Pre(v)(p))\otimes Post(v)(p)\end{aligned}$$

Proposition 8. *Let $(P,E,Pre,Post)$ be a net over a lifo Petri monoidal structure $\vartheta = (K,\sqsupseteq,\otimes,1)$. The following three statements are equivalent*

1. $M \sqsupseteq Pre(u) \;\; \wedge M' = M\diagup Pre(u)\otimes Post(u)$
2. $M' \sqsupseteq Post(u) \;\; \wedge M = M\diagup Post(u)\otimes Pre(u)$
3. $M \sqsupseteq Pre(u) \;\; \wedge M' \sqsupseteq Post(u) \;\; \wedge M\diagup Pre(u) = M\diagup Post(u)$

Proof.

$$\left(M' \sqsupseteq M\diagup Pre(u)\otimes Post(u)\right) \Leftrightarrow \left(M' \sqsupseteq Post(u) \;\wedge\; M\diagup Post(u) \sqsupseteq M\diagup Pre(u)\right)$$

together with

$$(M\diagup Pre(u)\otimes Post(u) \sqsupseteq M') \Leftrightarrow (M\diagup Post(u) \sqsupseteq M\diagup Pre(u))$$

establishes the equivalence (1) $\Leftrightarrow$ (3). Equivalence (2) $\Leftrightarrow$ (3) follows in the same manner.

The case of lifo nets is treated similarly.

We let $M\,[u\rangle\, M'$ when one of the three equivalent statements of the previous proposition is met and we say that the sequence $u \in E^*$ is enabled in marking M and leads to marking M'. We let $M[u\rangle \Leftrightarrow (\exists M'\; M\,[u\rangle\, M')$ and $[u\rangle M' \Leftrightarrow (\exists M\; M\,[u\rangle\, M')$

Proposition 9. $M[u\rangle \Leftrightarrow (M \sqsupseteq Pre(u))$ *and* $[u\rangle M' \Leftrightarrow (M' \sqsupseteq Post(u))$

Proof. By definition $M[\varepsilon\rangle M$ always holds. The firing relation of an event $e \in E$ is given by

$$M[e\rangle M' \Longleftrightarrow M \sqsupseteq Pre(e) \quad \wedge \; M' = M \diagup Pre(e) \otimes Post(e)$$

from which the equivalence $M[e\rangle \Leftrightarrow (M \sqsupseteq Pre(e))$ immediately follows. We then proceed by induction by showing that $M[uv\rangle \Leftrightarrow \exists M' \; M[u\rangle M' \; \wedge \; M'[v\rangle$:

$$\begin{aligned}
&M[uv\rangle \Leftrightarrow [M \sqsupseteq Pre(uv) \quad = (Pre(v) \diagup Post(u)) \otimes Pre(u)] \Leftrightarrow \\
&[M \sqsupseteq Pre(u) \; \wedge \; M \diagup Pre(u) \sqsupseteq Pre(v) \diagup Post(u)] \Leftrightarrow \\
&M[u\rangle M' \; \wedge \; M' = M \diagup Pre(u) \otimes Post(u) \sqsupseteq Pre(v) \diagup Post(u) \otimes Post(u) \sqsupseteq Pre(v) \\
&\Leftrightarrow [M[u\rangle M' \; \wedge \; M'[v\rangle]
\end{aligned}$$

The second equivalence relation is proved similarly.

6 Conclusion

We have suggested in this paper a definition of generalized Petri nets as nets enriched over certain closed monoidal structures. In the case of non commutative structures two semantics have been considered. The purpose of this paper was to identify algebraic structures that allow us to mimic the token game of Petri nets. However in order to be able to represent meaningful classes of nets, like for instance continuous Petri nets ([5],[6]), this work should be extended, and this can be done in several directions. First one should not necessarily consider that place contents and flow relations take their values in the same domain : one might consider that flow relations act on the place contents by making these values range in some module of which events act through flow relations. In the same manner one might also consider more complex trajectories than simply those indexed by words. Finally it would be necessary to be able to internalize the set of net computations in order to derive a duality between this enriched nets and corresponding enriched automata.

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