

Stochastic Ray Tracing

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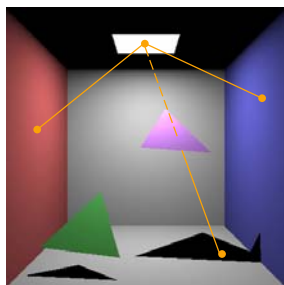
Contents

- Monte Carlo Integration
 - Application to direct lighting by area light sources
- Solving the radiance equation with the Monte Carlo Method
 - Distributed Ray Tracing
 - Path Tracing



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Classical Ray Tracing

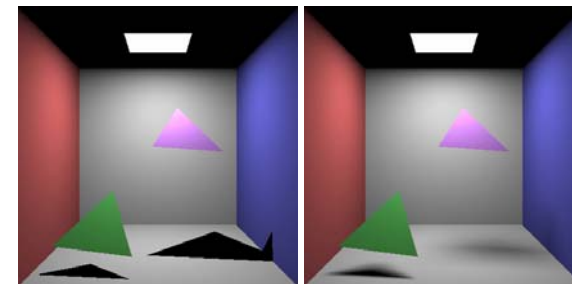


- One shadow ray by intersection point
- Only point light sources
- Hard shadows: umbra



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With Area Light Sources

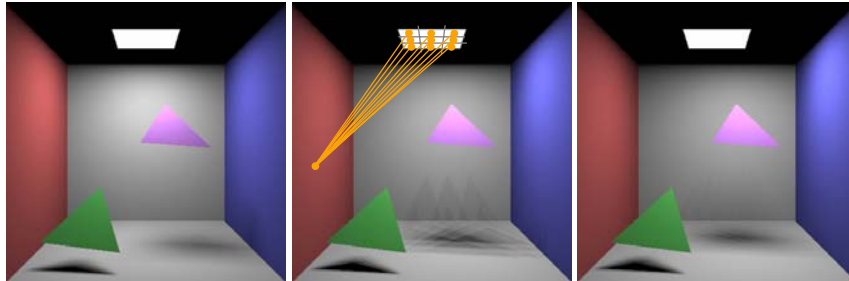


- Soft shadows
- Area light sources != point light sources



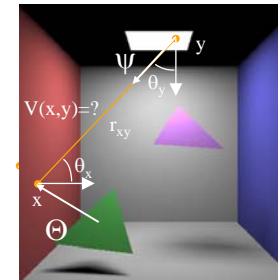
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More Sample Points on the Light Source



- Approximate the source by a set of points
- Aliasing along the shadows' borders

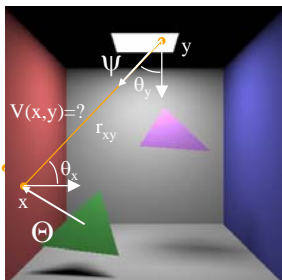
Solution to the Rendering Equation



- Analytical solution: too much difficult
- Use the Monte Carlo method

$$L(x \rightarrow \Theta) = \int_{AL} f_r(x, \Theta \leftrightarrow \Psi) L_e(y \rightarrow \Psi) V(x, y) \frac{\cos \theta_x \cos \theta_y}{r_{xy}^2} dy$$

Direct Lighting

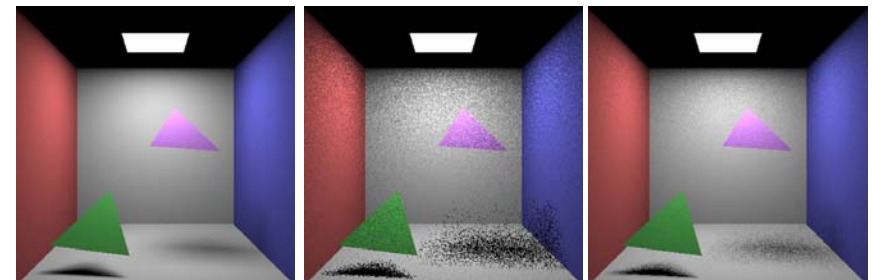


- Random point sampling of the area light sources
- Use these points to evaluate the integral

$$L(x \rightarrow \Theta) = \int_{AL} f_r(x, \Theta \leftrightarrow \Psi) L_e(y \rightarrow \Psi) V(x, y) \frac{\cos \theta_x \cos \theta_y}{r_{xy}^2} dy$$

$$L(x \rightarrow \Theta) = \int_{AL} D(y) dy \quad \langle L(x \rightarrow \Theta) \rangle = \frac{1}{N} \sum_{i=1}^N \frac{D(y_i)}{p(y_i)}$$

Direct Lighting



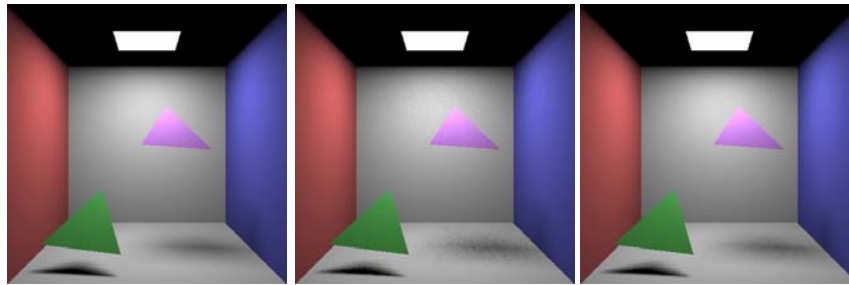
$$p(y) = \frac{1}{A_s}$$

1 shadow ray

9 shadow rays

$$\langle L(x \rightarrow \Theta) \rangle = \frac{1}{N} \sum_{i=1}^N A_s f_r(x, \Theta \leftrightarrow \Psi_i) L_e(y_i \rightarrow \Psi_i) \frac{\cos \theta_x \cos \theta_{y_i}}{r_{xy_i}^2} Vis(x, y_i)$$

Direct Lighting



36 shadow rays

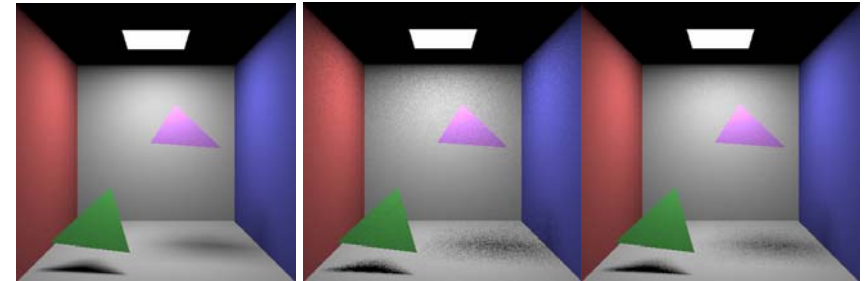
100 shadow rays

$$\langle L(x \rightarrow \Theta) \rangle = \frac{1}{N} \sum_{i=1}^N A_s f(x, \Theta \leftrightarrow \Psi) L_e(y_i \rightarrow \Psi_i) \frac{\cos \theta_x \cos \theta_{y_i}}{r_{xy}^2} \text{Vis}(x, y_i)$$



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Stratified Sampling



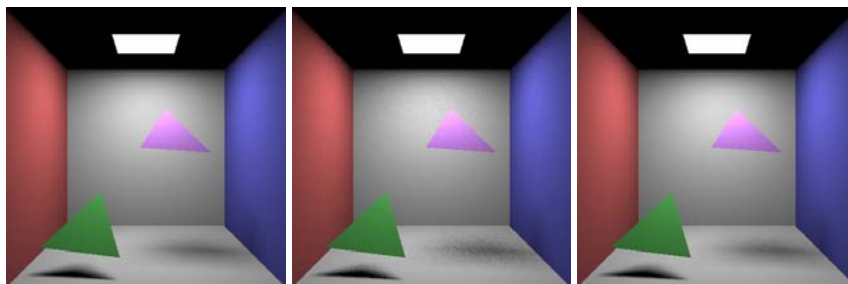
9 shadow rays
without stratification

9 shadow rays
with stratification



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Stratified Sampling



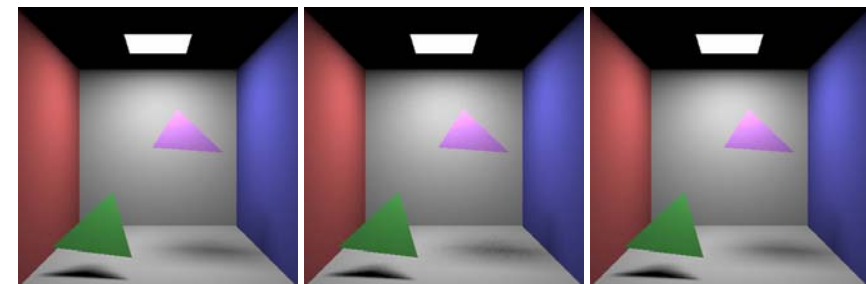
36 shadow rays no
stratification

36 shadow rays with
stratification



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Stratified Sampling



100 shadow rays no
stratification

100 shadow rays with
stratification



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Multiple Light Sources

- The integral does not change : rather than integrating over one light source, integrate over all the surfaces of the light sources

$$L(x \rightarrow \Theta) = \int_{A_L} f_r(x, \Theta \leftrightarrow \Psi) L_e(y \rightarrow \Psi) V(x, y) \frac{\cos \theta_x \cos \theta_y}{r_{xy}^2} dy$$

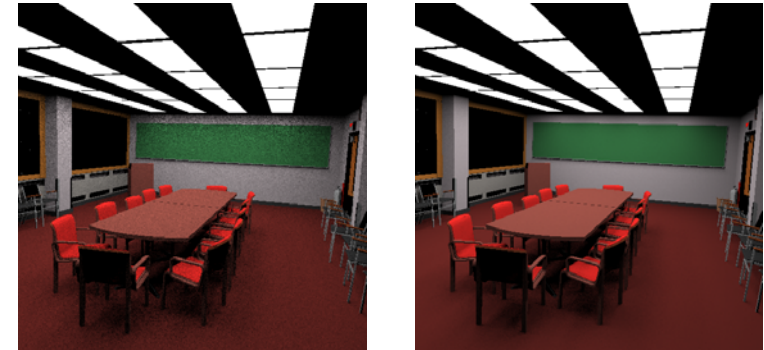
- The pdf for selecting points is modified : first select a light source S using pdf $p(S)$, then a point y on S with $p(y|S)$

$$p(S) = \frac{\Phi_S}{\Phi_T} \quad p(y|S) = \frac{1}{A_S} \quad p(y) = p(S)p(y|S)$$



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Multiple Light Sources

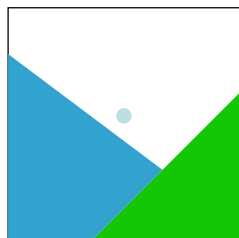


36 shadow rays per pixel in the 2 images but with different pdf



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Application to pixels: oversampling



- Compute radiance at the centre of a pixel \rightarrow aliasing
- Oversample a pixel and compute radiance for sub-pixels
- Use a filter

$$L = \int_{Pixel} f(x) L(x) dx$$

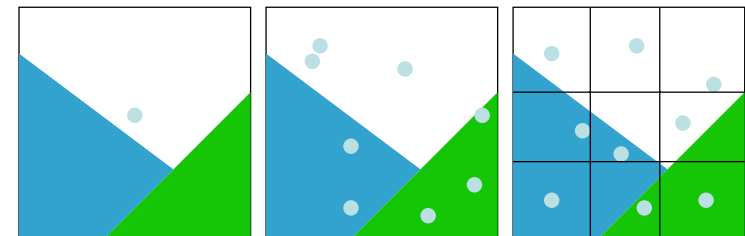
... evaluated by the Monte Carlo method.



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Application to pixels: oversampling

- Any sampling method

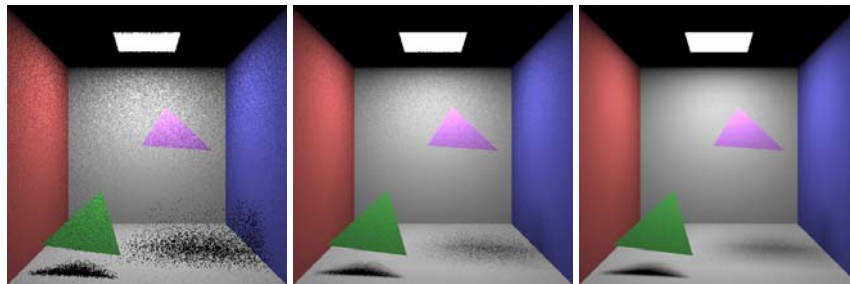


$$L \approx \frac{1}{N} \sum_{i=1}^N \frac{f(x_i) L(x_i)}{p(x_i)}$$



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Application to pixels: oversampling



1 ray / pixel

10 rays / pixel

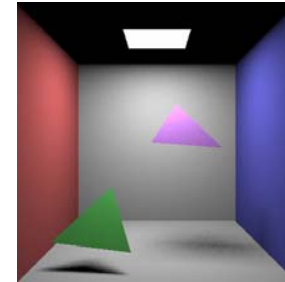
100 rays / pixel



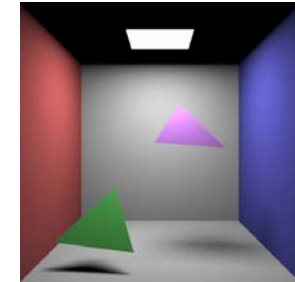
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Implementation

- Comparison :



1 centered ray per pixel
100 random shadow rays
per intersection



100 centered rays per pixel
1 random shadow rays per
intersection



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The Rendering Equation

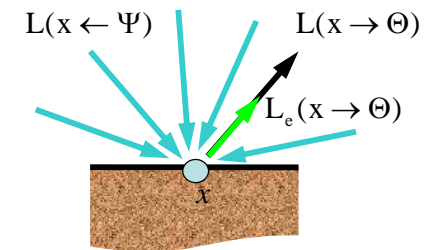
- Evaluation of the rendering equation
- How to write the rendering equation and how to evaluate it using Monte Carlo integration?
 - Which pdf to use for the rendering equation ?
 - Algorithms and results



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The Rendering Equation

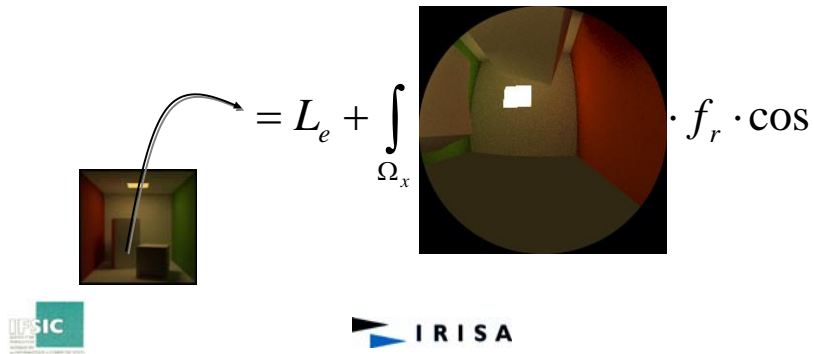
$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$



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The Rendering Equation

$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + \int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$

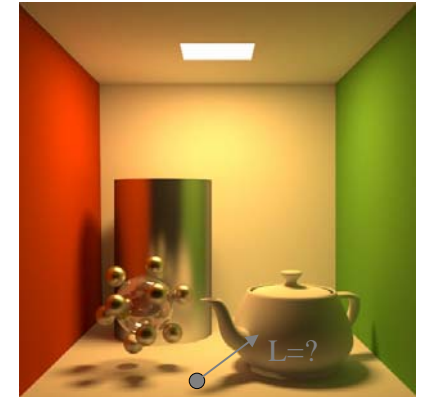


Computing Radiance

- How to evaluate L ?

– Find $L_e(x \rightarrow \Theta)$

– Add:



$$\int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$

Computing Radiance

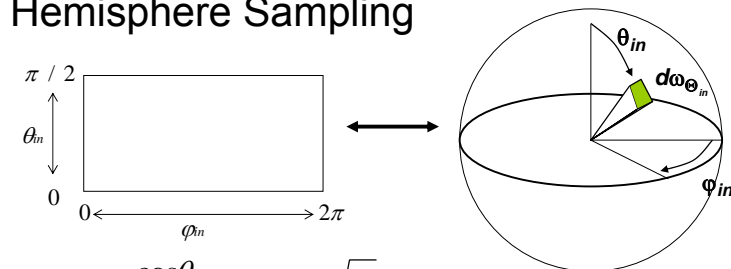
- How to evaluate L ?
 - Monte Carlo Integration
 - Generate random directions on Ω_x , using the probability density function $p(\Psi)$

$$L(x \rightarrow \Theta) = \int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$

$$\langle L(x \rightarrow \Theta) \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f_r(x, \Psi_i \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi_i) \cdot \cos(\Psi_i, n_x)}{p(\Psi_i)}$$

Computing Radiance

- Hemisphere Sampling



$$p(\Theta) = \frac{\cos \theta}{\pi} \Rightarrow \theta = \arccos(\sqrt{\xi_1}) \text{ et } \varphi = 2\pi \xi_2$$

$$\Theta = (\theta, \varphi) \Rightarrow d\omega = \sin \theta \cdot d\theta \cdot d\varphi$$

$$p(\Theta) = \frac{n+1}{2\pi} \cos^n \theta \Rightarrow \theta = \arccos(\xi_1^{\frac{1}{n+1}}) \text{ et } \varphi = 2\pi \xi_2$$

Computing Radiance

- Generate a random direction Ψ_i

$$\langle L \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f_r(\dots) \cdot L(x \leftarrow \Psi_i) \cdot \cos(\dots)}{p(\Psi_i)}$$

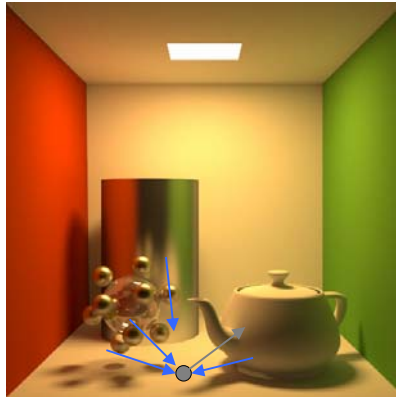
Evaluate the BRDF

Evaluate the $\cos(\dots)$

Evaluate $L(x \leftarrow \Psi_i)$



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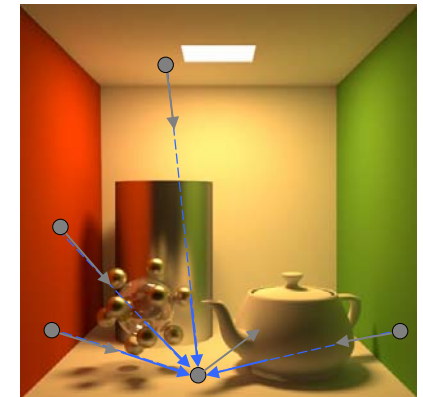


Computing Radiance

- Evaluation of $L(x \leftarrow \Psi_i)$?
- Radiance is constant along the propagation direction.
- $rc(x, \Psi_i)$ = first visible point.
- $L(x \leftarrow \Psi_i) = L(rc(x, \Psi_i) \rightarrow \Psi_i)$



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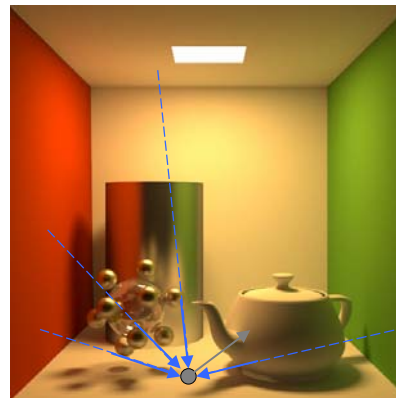


Computing Radiance

- Recursive Evaluation
- Each bounce adds a level of indirect lighting.

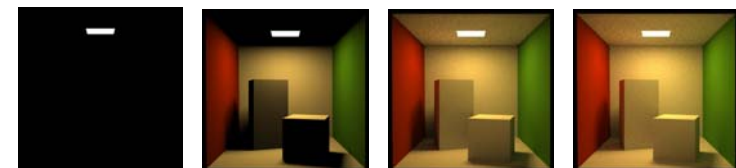


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Stopping Recursion

- When recursion is stopped?



The contributions of higher order reflections are negligible.

If we ignore them, the estimates are biased !



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Terminating the Recursion

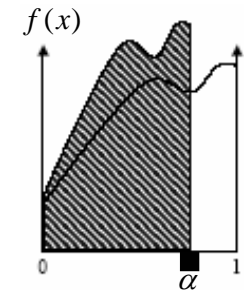
- When/how do we stop the recursion?
 - When the ray doesn't hit any object
 - Can be very hard/impossible for dense scenes
 - When a maximum depth is reached
 - This is highly scene dependent
 - Primarily specular scenes requires far more bounces than diffuse scenes
 - Having a fixed path length results in a biased estimate
 - When the contribution of the ray falls below a certain threshold
 - More efficient than a fixed max depth, but still gives a biased result

Russian Roulette

- Integral

$$I = \int_0^1 f(x) dx = \frac{1}{\alpha} \int_0^{\alpha} f(x/\alpha) dx$$
- Estimator

$$\langle I_{\text{roulette}} \rangle = \begin{cases} \frac{1}{\alpha} f(x/\alpha) & \text{if } x < \alpha \\ 0 & \text{if } x > \alpha \end{cases}$$
- Use Russian roulette to decide
 - ray is absorbed with probability $1-\alpha$
 - results in unbiased estimator



Russian Roulette

- A simple and unbiased termination criteria is *Russian roulette*:
 - Given a uniform random number ξ , terminate the ray if $\xi \geq \alpha$, otherwise scale the contribution of the ray by $1/\alpha$
 - Here $\alpha \in [0,1]$ is the absorption probability
 - Recursion stops with a probability of $p = 1 - \alpha$
 - By scaling the contribution of rays that continue by $1/\alpha$, the result remains unbiased
- Russian roulette is not practically useful until we add direct light to our ray tracer!!

Russian Roulette

- Example
 - $p = 0.9$, then $\alpha = 1 - p = 0.1$
 - One chance in 10 that ray is reflected.
 - The radiance due to one reflected ray is multiplied by 10.
 - instead of shooting 10 rays, we shoot only 1, but count the contribution of this one 10 times

Russian Roulette

- Case of n incident rays

$$\langle L(x \rightarrow \Theta) \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f_r(x, \Psi_i \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi_i) \cdot \cos(\Psi_i, n_x)}{p(\Psi_i)}$$

- Case of one incident ray

$$\begin{aligned} \langle L(x \rightarrow \Theta) \rangle &= \frac{f_r(x, Di \leftrightarrow \Theta) \cdot L(x \leftarrow Di) \cdot \cos(Di, n_x)}{p(Di)} \\ &= R(x, Di) \cdot L(x \leftarrow Di) \end{aligned}$$



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Russian Roulette

- With Russian roulette the pseudo code now looks like this:

```

RGB radiance(Ray r)
  if(r hits at x)
    if( $\xi < \alpha$ )
      Generate new direction, Di, from  $p(\Psi_i)$  and
      the surface normal at x
      Ray ray(x, Di)
      return  $L_e(x) + R(x, Di) \cdot \text{radiance}(\text{ray}) / \alpha$ 
    else
      return  $L_e(x)$ 
  else
    return background
  
```



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Russian Roulette

- Non biased Estimate
- The expected value is correct
- Bigger variance
- But more efficient



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Distribution Ray Tracing

- Algorithm
 - Trace N rays per pixel
 - At each intersection point, trace 1 ray (or more) randomly chosen on the hemisphere to evaluate the rendering equation
 - End recursion using the Russian roulette



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Distribution Ray Tracing

```

computeImage() {
  for each pixel (i,j) {
    estimatedRadiance[i,j] = 0
    for s = 1 to #samples-in-pixel {
      generate Q in pixel (i,j)
      theta = (Q - E)/|Q-E| // E is the Eye
      x = trace(E,theta)
      estimatedRadiance [i,j] += computeRadiance(x,-theta)
    }
    estimatedRadiance [i,j] /= # samples-in-pixel
  }
}

computeRadiance(x, theta) {
  estimatedRadiance = basicPT(x, theta)
  return estimatedRadiance
}
    
```



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Distribution Ray Tracing

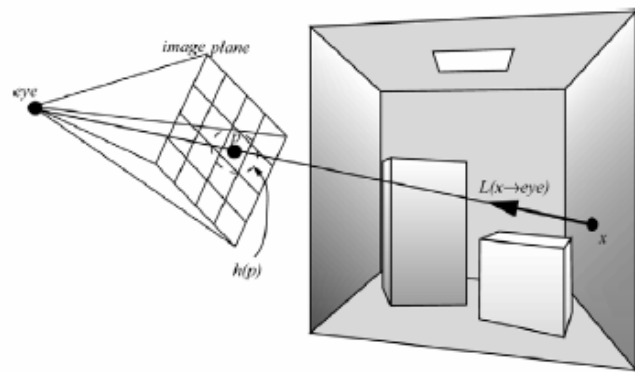
```

basicPT(x, theta) {
  estimatedRadiance = Le(x, theta)
  if(not absorbed) { // russian roulette
    for s = 1 to #radianceSamples { // ray directions
      psi = generate random direction on hemisphere
      y = trace(x, psi)
      estimatedRadiance +=
        basicPT(y,-psi) * BRDF(x,psi,theta)*
        cos(Nx,psi) / pdf(psi)
    }
    estimatedRadiance /= #radianceSamples
  }
  return estimatedRadiance/(absorption)
}
    
```



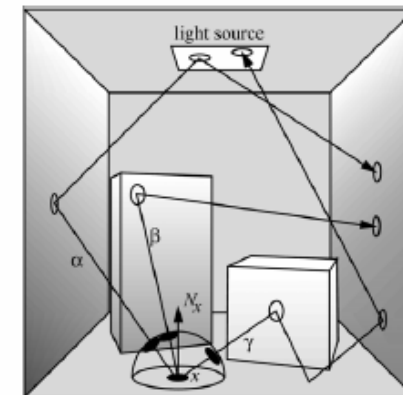
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Distribution Ray Tracing



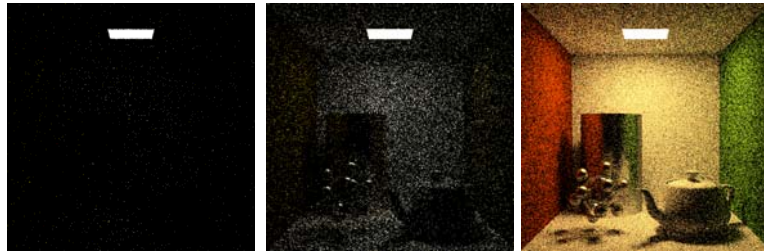
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Distribution Ray Tracing



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Distribution Ray Tracing



1 ray/pixel

16 rays/pixel

256 rays/pixel

Very noisy : null contribution as long as the path does not reach a light source!!



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Distribution Ray Tracing



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Distribution Ray Tracing



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Distribution Ray Tracing



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Distribution Ray Tracing



Distribution Ray Tracing



Distribution Ray Tracing



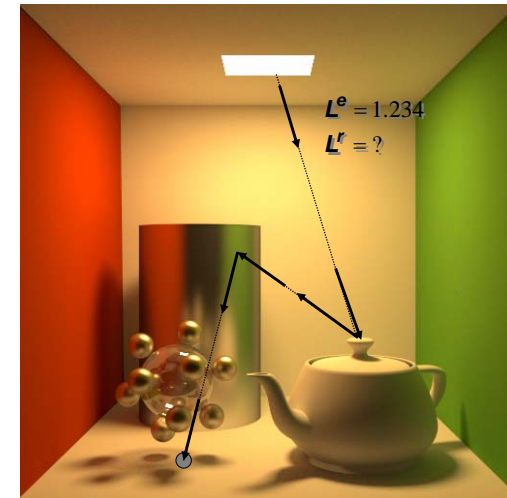
Distribution Ray Tracing



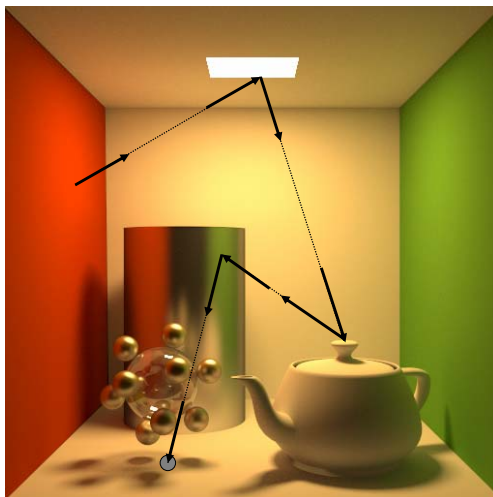
Distribution Ray Tracing



Distribution Ray Tracing



Distribution Ray Tracing



Distribution Ray Tracing

- Improve the algorithm by dividing the integral into two parts: direct and indirect

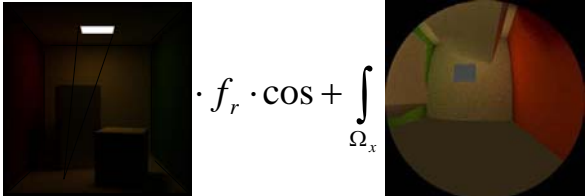
$$L_i(x \rightarrow \Theta) = \int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L_e(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$

$$+ \int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L_i(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$

$$L(x \rightarrow \Theta) = L_e(x \rightarrow \Theta) + L_i(x \rightarrow \Theta)$$

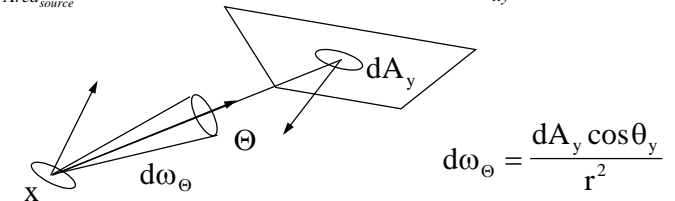
Distribution Ray Tracing

- Evaluate differently the direct and the indirect components

$$\int_{Source} \cdot f_r \cdot \cos + \int_{\Omega_x} \cdot f_r \cdot \cos$$


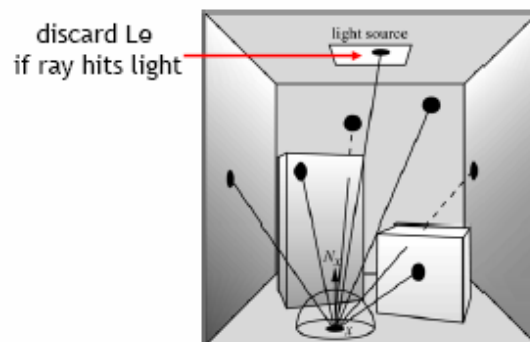
Direct Illumination

$$\begin{aligned} L_{direct}(x \rightarrow \Theta) &= \int_{\Omega_x} f_r(\dots) \cdot L_e(x \leftarrow \Psi) \cdot \cos(\dots) \cdot d\omega_\Psi \\ &= \int_{\Omega_x} f_r(\dots) \cdot L_e(y \rightarrow x) \cdot \cos(\dots) \cdot d\omega_\Psi \\ &= \int_{Area_{source}} f_r(\dots) \cdot L_e(y \rightarrow x) \cdot \cos(\dots) \cdot \left(\frac{\cos(\dots) dA_y}{r_{xy}^2} \right) \cdot V(x, y) \end{aligned}$$



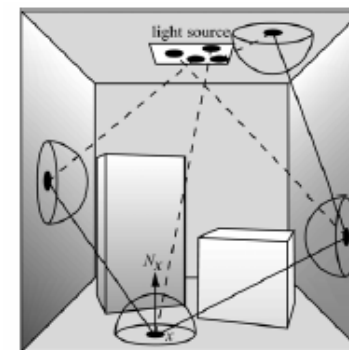
Indirect Illumination

$$L_i(x \rightarrow \Theta) = \int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L_i(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$



Indirect Illumination

$$L_i(x \rightarrow \Theta) = \int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L_i(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$



Distribution Ray Tracing

$$L(x \rightarrow \Theta) = \int_{\Omega_x} f_r(x, \Psi \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi) \cdot \cos(\Psi, n_x) \cdot d\omega_\Psi$$

$$\langle L(x \rightarrow \Theta) \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f_r(x, \Psi_i \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi_i) \cdot \cos(\Psi_i, n_x)}{p(\Psi_i)}$$

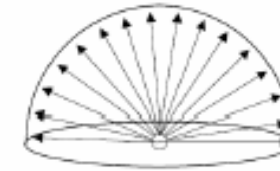
- Depends on how to sample the hemisphere
 - Uniform distribution
 - Importance sampling : pick p to match integral
 - Cosine distribution
 - BRDF distribution
 - BRDF*cosine distribution

Distribution Ray Tracing

•Uniform distribution

$$p(\Psi) = \frac{1}{2\pi}$$

$$\langle L(x \rightarrow \Theta) \rangle = \frac{2\pi}{N} \sum_{i=1}^N f_r(x, \Psi_i \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi_i) \cdot \cos(\Psi_i, n_x)$$

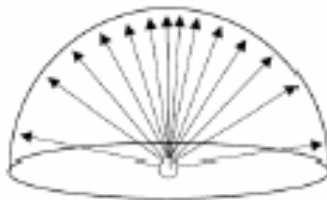


Distribution Ray Tracing

•Cosine distribution

$$p(\Psi) = \frac{\cos \theta_\Psi}{\pi}$$

$$\langle L(x \rightarrow \Theta) \rangle = \frac{\pi}{N} \sum_{i=1}^N f_r(x, \Psi_i \leftrightarrow \Theta) \cdot L(x \leftarrow \Psi_i)$$



Distribution Ray Tracing

•BRDF distribution

$$p(\Psi) = f_r(\dots)$$

$$\langle L(x \rightarrow \Theta) \rangle = \frac{1}{N} \sum_{i=1}^N L(x \leftarrow \Psi_i) \cdot \cos(\Psi_i, n_x)$$

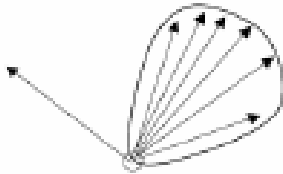


Distribution Ray Tracing

•BRDF*cosine distribution

$$p(\Psi) = f_r(\dots) \cdot \cos \theta_\Psi$$

$$\langle L(x \rightarrow \Theta) \rangle = \frac{1}{N} \sum_{i=1}^N L(x \leftarrow \Psi_i)$$



Distribution Ray Tracing: pixel sampling

```
computeImage() {
  for (each pixel (i,j)) {
    estimatedRadiance[i,j] = 0
    for (s = 1 to #samples-in-pixel) {
      generate Q in pixel (i,j)
      theta = (Q - E)/|Q-E|
      x = trace(E,theta)
      estimatedRadiance [i,j] +=
        computeRadiance(x,-theta)
    }
    estimatedRadiance [i,j] /= #samples-in-pixel
  }
}
```

Distribution Ray Tracing: radiance estimation

```
computeRadiance(x,theta) {
  estimatedRadiance = Le(x,theta)
  estimatedRadiance += directIllumination(x, theta)
  estimatedRadiance += indirectIllumination(x, theta)
  return estimatedRadiance
}
```

Distribution Ray Tracing: direct illumination

```
directIllumination(x,theta) {
  estimatedRadiance = 0
  for (s = 1 to #shadowRays) {
    k = pick random light
    y = generate random point on light k
    psi = (x-y) / |x-y|
    estimatedRadiance += Le_k(y,-psi) *
      BRDF(x,psi,tetha) * G(x,y) * V(x,y) /(p(k)*p(y|k))
  }
  estimateRadiance /= #shadowRays
  return estimatedRadiance
}
```

Distribution Ray Tracing: direct illumination

```

directIllumination(x,theta) {
    estimatedRadiance = 0
    for (k=1 to #lights) {
        for (s = 1 to #shadowRays) {
            y = generate random point on light k
            psi = (x-y) / |x-y|
            estimatedRadiance += Le_k(y,-psi) *
                BRDF(x,psi,theta) * G(x,y) * V(x,y) / p(y)
        }
    }
    estimateRadiance /= #shadowRays
    return estimatedRadiance
}

```



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Distribution Ray Tracing: indirect illumination

```

indirectIllumination(x,theta) {
    estimatedRadiance = 0
    if (not absorbed) { // russian roulette
        for (s = 1 to #indirectDirectionSamples) {
            psi = generate random direction on hemisphere
            y = trace(x, psi)
            estimatedRadiance += computeRadiance(y,-psi) *
                BRDF(x,psi,theta) * cos(Nx,psi) / pdf(psi)
        }
        estimatedRadiance /= #indirectDirectionSamples
    }
    return estimatedRadiance /(absorption)
}

```



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Distribution Ray Tracing

- To sum up
 - For primary rays :
 - use many ray samples at the intersection point
 - Use uniform or cosine pdf to sample the hemisphere
 - For shadow rays :
 - Use uniform area-based pdf to sample the light sources
 - Use many samples
 - For secondary rays
 - Use one or more samples
 - Use BRDF based pdf



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Tracing Lambertian Materials

- We've already seen that for Lambertian materials, R is just a constant between 0 and 1
- To generate a random ray direction, we use the cosine density $p(\theta, \phi) = \frac{1}{\pi} \cos \theta$ and two uniform random numbers ξ_1 and ξ_2
 - Using the techniques presented before, we find that

$$\cos \theta = \sqrt{1 - \xi_1}$$

$$\phi = 2\pi\xi_2$$



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Imperfect Specular Reflections

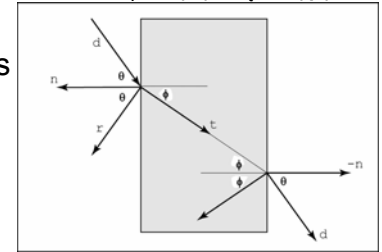
- Perfectly reflecting materials are rare
 - Usually, the reflection is slightly blurred
- To achieve an imperfect specular reflection, we can choose the reflection direction from a phong density:

$$p(\mathbf{k}) = \frac{m+1}{2\pi} (\mathbf{r} \cdot \mathbf{k})^m \Rightarrow \begin{aligned} \cos\theta &= (1 - \xi_1)^{\frac{1}{m+1}} \\ \phi &= 2\pi\xi_2 \end{aligned}$$

- Where \mathbf{r} is the mirror reflection direction and θ is the angle between \mathbf{r} and \mathbf{k}

Transparent Materials

- When a ray hits a transparent material, it is either reflected or transmitted
 - When a ray is transmitted from a medium with refractive index n_i to a medium with refractive index n_t , it is bend according to Snell's law: $n_i \sin(\theta) = n_t \sin(\phi)$
 - For an incident angle of θ , the fraction of incident rays that are reflected is $R(\theta)$. $1-R(\theta)$ is the fraction of transmitted rays



Path tracing

- At each intersection point one can make a choice
 - Reflection or refraction?
 - If reflection : diffuse or specular?

Reflection or Transmission?

- When striking a transparent surface, we need to make a choice: Should the new ray be a reflected or transmitted ray?
- We can set a transmission probability P , and then pick a random number ξ .
 - If $\xi < P$, the ray is transmitted, and the contribution is scaled by $1/P$
 - Else, the ray is reflected, and the contribution is scaled by $1/(1-P)$

Beer's Law

- When light travels through an 'impure' medium, its radiance is attenuated according to Beer's law:

$$I(s) = I(0)e^{\ln(a)s}$$

- Here $I(s)$ is the radiance of a ray at a distance s from the interface and a is the RGB attenuation constant

Specular-Diffuse Surfaces

- Most surfaces reflect light as some combination of specular and diffuse reflections
 - When the angle between the view vector and the normal increases, the specular reflection increases and the diffuse decreases
- We model such materials by linearly combining a specular and a diffuse material

Specular-Diffuse Surfaces

- We can choose the specular ray with probability P and the diffuse ray with probability $1-P$

if($\xi < P$)

return $R(x, D_i) * \text{radiance}(\text{specular ray}) / P$

else

return $R(x) * \text{radiance}(\text{diffuse ray}) / (1-P)$

Results



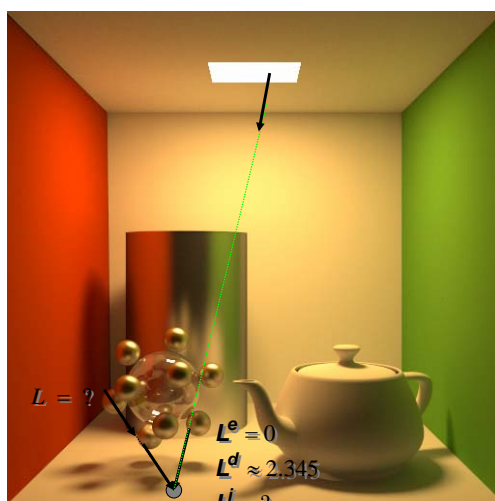
Results



Results



Results



Results



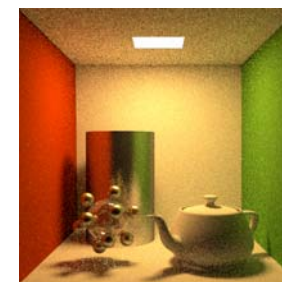
Results



Comparison



Without computing
direct lighting

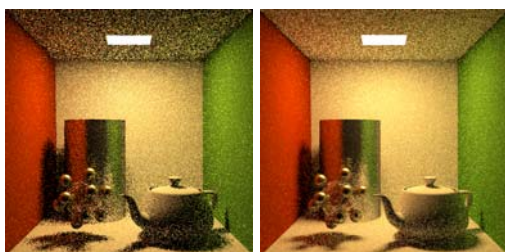


With direct lighting
computation

16 rays/pixel

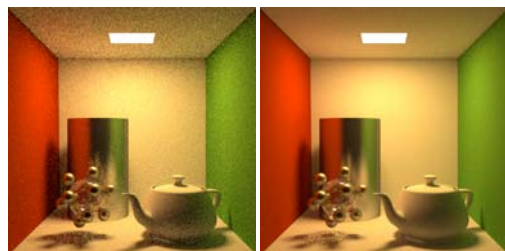
Comparison

1 ray/
pixel



4 rays/
pixel

16 rays/
pixel



256 rays/
pixel