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COMPUTER-AIDED GEOMETRIC DESIGN AND COMPUTER GRAPHICS:

B-SPLINES AND NURBS CURVES AND SURFACES

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B-splines Curves

Knot vector: a list $t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_{m-1} \leq t_m$ of $m+1$ nondecreasing numbers, such that the same value should not appear more than k times, (k =order of the B-spline).

We define the i-th B-spline function $N_{ik}(t)$ of order k (= $k-1$ degree) as:

$$N_{i1}(t) = \begin{cases} 1 & \textit{if} \quad t_i \leq t \leq t_{i+1} \\ 0 & \textit{otherwise} \end{cases}$$

$k=1$

$$N_{ik}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

$k>1$

1. $N_{ik}(t) > 0 \quad \textit{for} \quad t_i < t < t_{i+k}$
2. $N_{ik}(t) = 0 \quad \textit{for} \quad t_0 \leq t \leq t_i,\ t_{i+k} \leq t \leq t_{n+k}$
3. $\sum_{i=0}^{n} N_{ik}(t) = 1 \quad t \in [t_{k-1}, t_{n+1}]$ Normalizing property

B-splines Curves

Given a set of $n+1$ control points (called de Boor points) d_i $(i=0,....,n)$ and a knot vector $T=[t_0,t_1,...,t_{m-1},t_m]$ we define a B-spline $\mathbf{X}(t)$ of order k as:

$$\mathbf{X}(t) = \sum_{i=0}^{n} \mathbf{d}_i N_{ik}(t)$$

where $N_{ik}(t)$ describes the blending B-spline function of degree k-1 associated with the knot vector T.

PROPERTIES:

(a) The degree of the polynomial does not exceed k-1.

(b) The first k-2 derivatives are continuous.

Knot vector

$m+1 = (n+1) + k$

# of knots — # of control points — order of curve

$$m = n + k$$

Knot vectors can be classified as:

Periodic / Uniform → $t_i - t_{i-1} = C$

B-splines functions are all translates of each other: n=3, k=3 [0 1 2 3 4 5 6]

NonPeriodic → k repeated values at the ends (k = order)

n=3, k=3 [0 0 0 1 2 2 2]

NonUniform → **NURBS**

B-splines Curves

Knot vector

Periodic / Uniform

• Influence of each basis function is limited to *k* intervals

• Parameter range: $(k-1) \leq t \leq (n+1)$

EXAMPLE [0 1 2 3 4 5 6]

NonPeriodic

• No loss of parameter range: the curve interpolates the first and the last control points

• Parameter range: $0 \leq t \leq n-k+2$

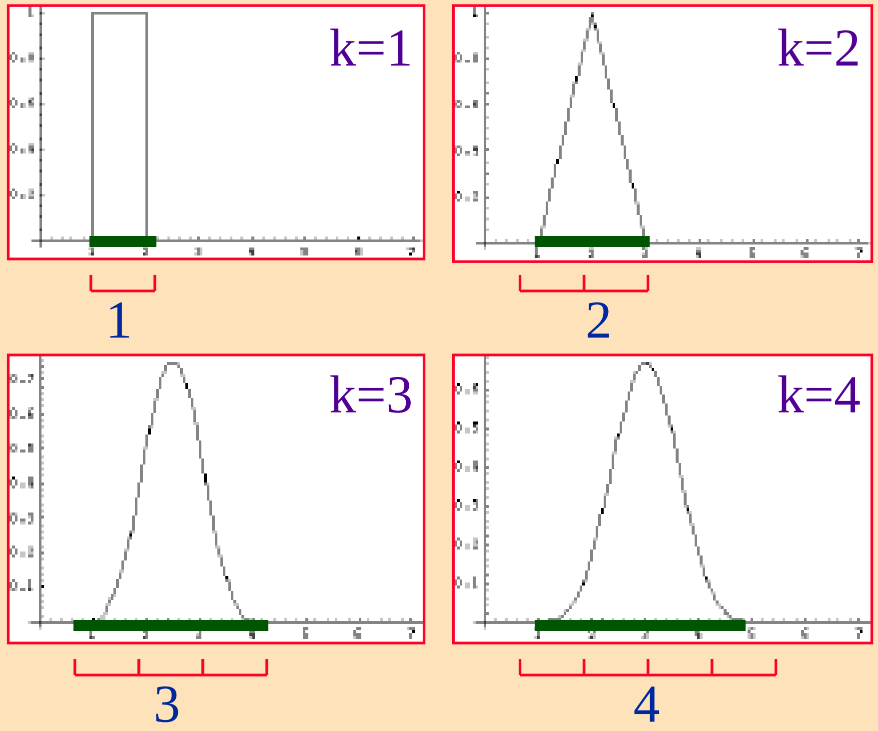
EXAMPLE [0 0 0 1 2 2 2]

NonUniform (NURBS)

EXAMPLES [0 1 2 3 3 3 4]

[0 0 0 1 2 5 5]

EXAMPLE *B-splines* $B_{2k}(t)$



Bézier representation

It is a simple case of B-splines, when:

• # of control points = order B-spline

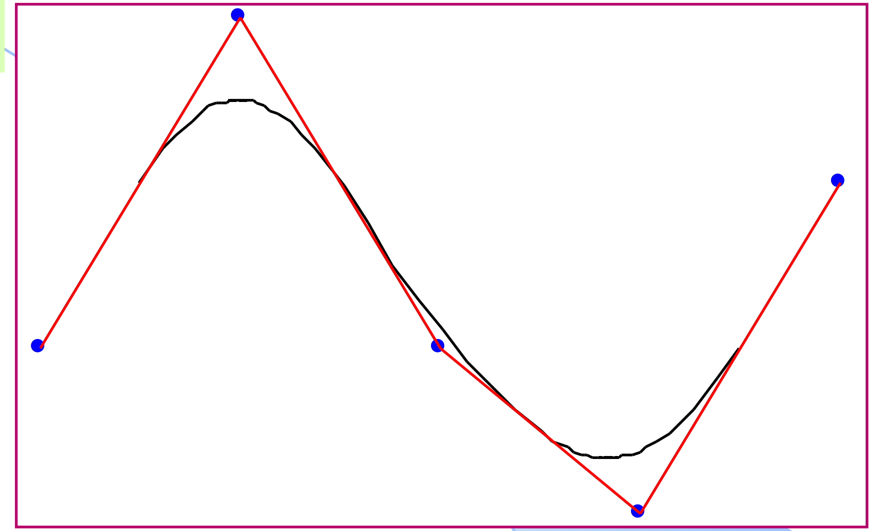
• A nonperiodic knot vector is considered

B-splines Curves

Example: Five control points $n=4$

Quadratic B-spline: $k=3$

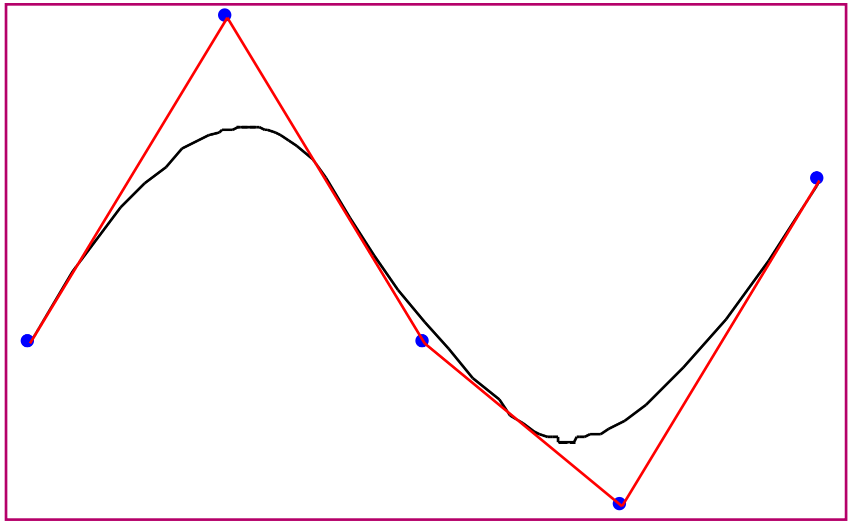
$P=((1,2),(2,4),(3,2),(4,1),(5,3))$



Periodic / Uniform $T=[0\ 1\ 2\ 3\ 4\ 5\ 6\ 7]$

Parameter range: (2,5)

Curve *does not interpolate* the end points.



NonPeriodic $T=[0\ 0\ 0\ 1\ 2\ 3\ 3\ 3]$

Parameter range: (0,3)

Curve *interpolates* the end points.

For the case $n=k$, we recover the *Bézier curves*.

NonUniform

$T=[0\ 1\ 2\ 3\ 4\ 4\ 4\ 5]$

$T=[0\ 2\ 2\ 5\ 5\ 5\ 26]$

$T=[0\ 0\ 0\ 1\ 2\ 5\ 6\ 6]$

$T=[0.3\ 2\ 2\ 8\ 8\ 9.6\ 9.6]$

Rational B-splines Curves

Introducing weights w_{ij} we obtain a rational B-spline curve given by:

$$R(t) = \frac{\sum_{i=0}^{n} P_i \; w_i \; N_{ik}(t)}{\sum_{i=0}^{n} w_i \; N_{ik}(t)}$$

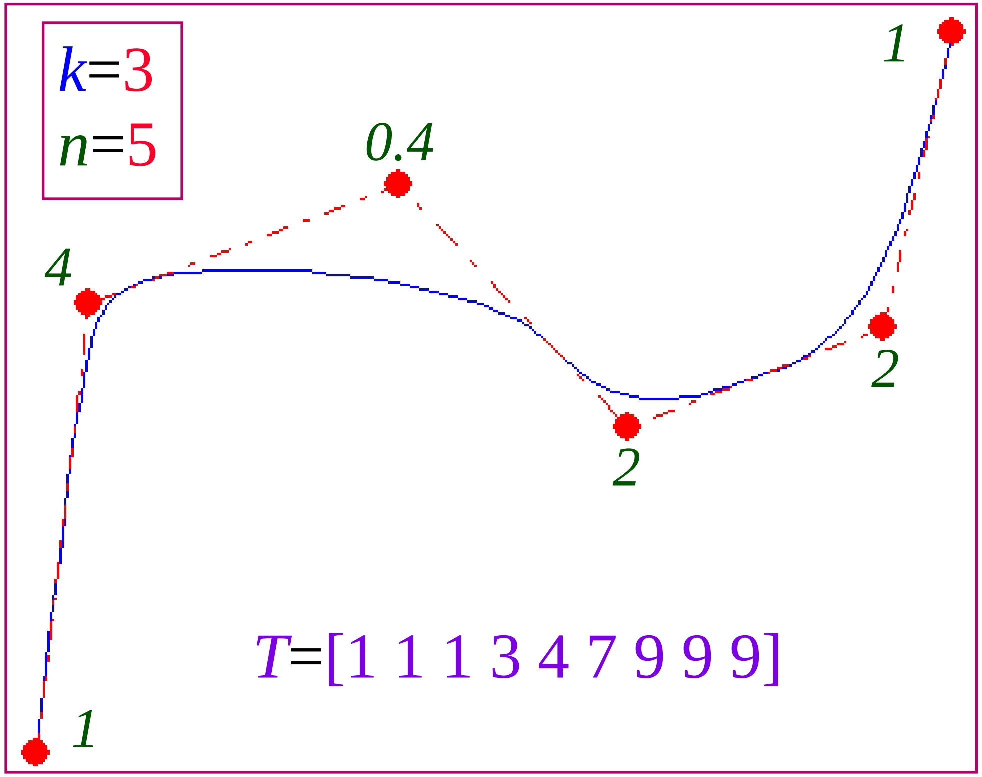
Example: A full circle can be obtained by using seven control points:

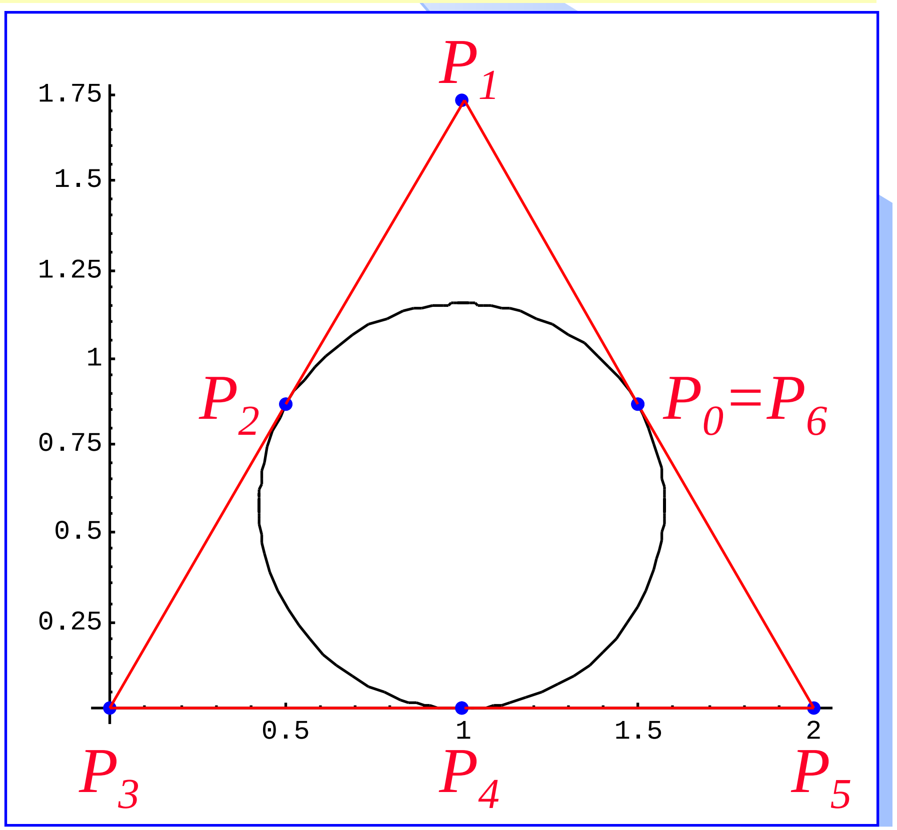
$\{P_0, P_1, P_2, P_3, P_4, P_5, P_6\}$

the knot vector:

$T=[0\ 0\ 0\ 1/3\ 1/3\ 2/3\ 2/3\ 1\ 1\ 1]$

and weights: $w=\{1,1/2,1,1/2,1,1/2,1\}$



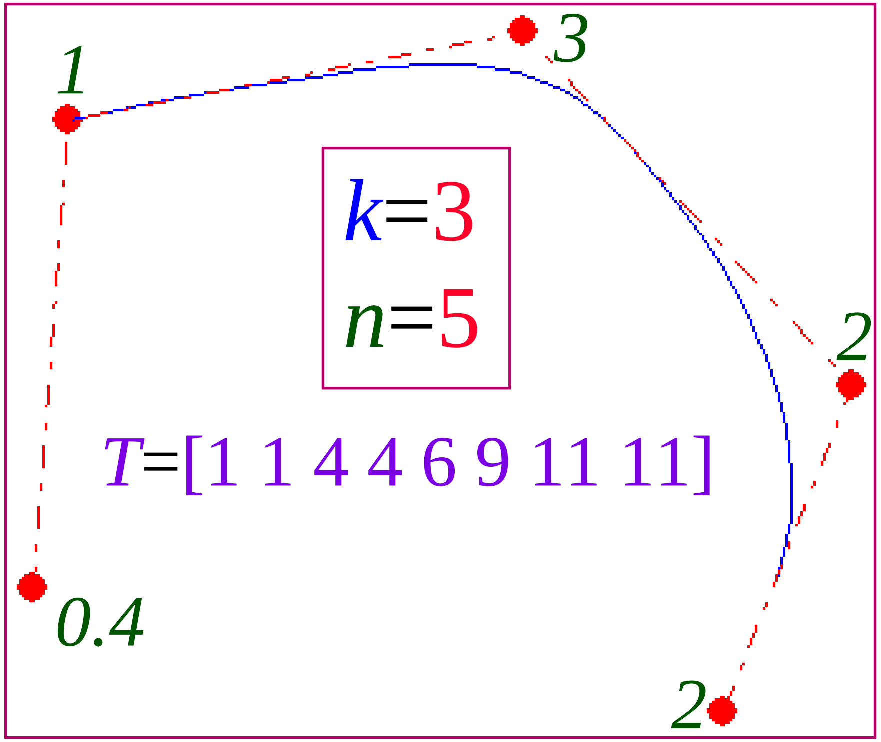


Non-Uniform Rational B-splines (NURBS)

Today, NURBS become the most important geometric entity in design.

Incorporated in the most popular CAD/CAM and Computer Graphics systems

A NURBS curve with no interior knots is a rational Bézier curve. So, *NURBS curves contain nonrational B-splines and rational and nonrational Bézier curves* as special cases.



Unfortunately, it is no easy to understand how they work

Non *Uniform* *Rational* *B-* *Splines*	*@*	*Nobody* *Understands* *Rational* *B-* *Splines*

Piegl, L. and Tiller, W: *The NURBS Book*, 2nd. Edition, Springer Verlag Berlin, 1997.

B-splines Surfaces

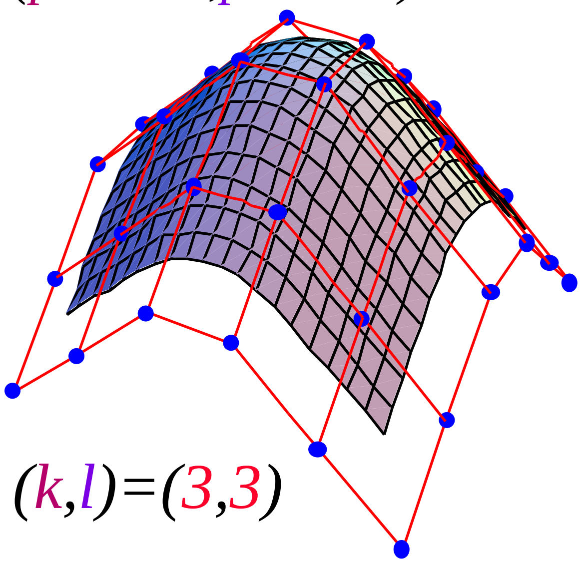
A B-spline surface S of order k in the u direction and order l in the v direction is a bivariate vector-valued piecewise function of the form:

$$S(u,v)=\sum_{j=0}^{m}\sum_{i=0}^{n} P_{ij}\,N_{ik}(u)N_{jl}(v)$$

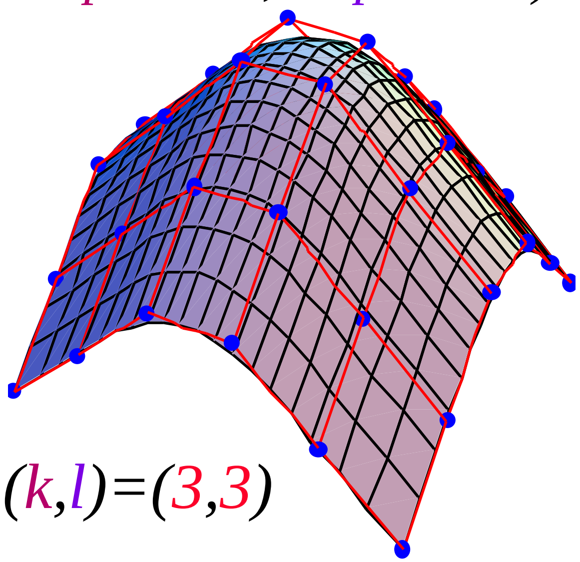
where the P_{ij} form a bidirectional net and the $N_{ik}(u)$ and $N_{jl}(v)$ are the B-spline functions defined on the knot vectors:

$U=[u_0, u_1,\ldots, u_{p-1}, u_p]$ and $V=[v_0, v_1,\ldots, v_{q-1}, v_q]$ $(p = m+k,\ q = l+n)$

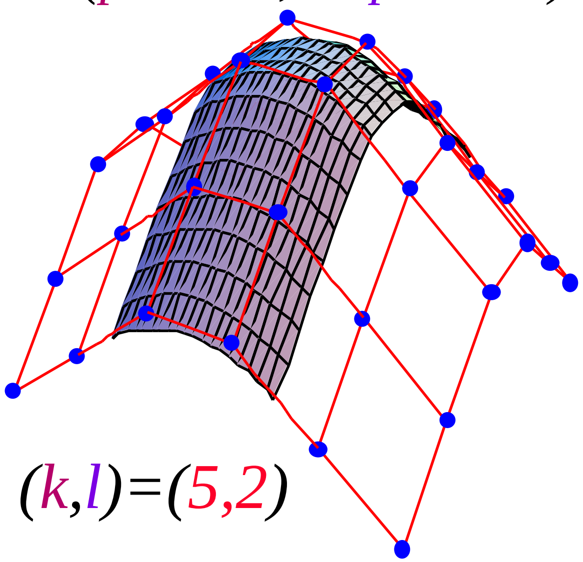
(periodic,periodic)



(nonperiodic,nonperiodic)



(periodic,nonperiodic)



Properties of the B-splines Surfaces

1. If $U=[\overbrace{0,\ldots,0}^{k+1},\overbrace{1,\ldots,1}^{k+1}]$ and $V=[\overbrace{0,\ldots,0}^{l+1},\overbrace{1,\ldots,1}^{l+1}]$ then the surface S *interpolates the four corner points*:

$S(0,0)=P_{00}$, $S(1,0)=P_{m0}$, $S(0,1)=P_{0n}$, $S(1,1)=P_{mn}$

2. If $m=k$, $n=l$, $U=[\underbrace{0,\ldots,0}_{k+1},\underbrace{1,\ldots,1}_{k+1}]$ and $V=[\underbrace{0,\ldots,0}_{l+1},\underbrace{1,\ldots,1}_{l+1}]$ then S is a *Bézier surface*.

3. *Affine invariance:* an affine transformation is applied to the surface by applying it to the control points.

4. *Strong convex hull property*: if (u,v) belongs to $[u_r, u_{r+1})\mathbf{x}[v_s, v_{s+1})$, S is in the convex hull of the control points P_{ij} $(r-k<=i<r,\ s-l<=j<s)$

5. *Local modification scheme:* if P_{ij} is moved, it affects the surface only in the rectangle $[u_i, u_{i+k+1})\mathbf{x}[v_j, v_{j+l+1})$.

6. Isoparametric curves for S behave analogously to that the Bézier surfaces.

Rational B-splines Surfaces

A rational B-spline surface S of order k in the u direction and order l in the v direction is defined by:

$$S(u,v)=\frac{\sum_{j=0}^{m}\sum_{i=0}^{n} P_{ij}\; w_{ij}\; N_{ik}(u) N_{jl}(v)}{\sum_{j=0}^{m}\sum_{i=0}^{n} w_{ij}\; N_{ik}(u) N_{jl}(v)}$$

where the P_{ij}, $N_{ik}(u)$ and $N_{jl}(v)$ are given as for the nonrational case, and w_{ij} are the weights, which will be assumed $w_{ij}>0$, for all i,j.

Once again, if all $w_{ij}=1$, we recover the nonrational B-spline surfaces.

NURBS surfaces are included here as a particular case. However, some authors *identify rational B-spline surfaces to NURBS surfaces*. From this point of view, NURBS surfaces are a key topic in computer graphics, being incorporated in most of the computer design systems.

Piegl, L. and Tiller, W: *The NURBS Book*, 2nd. Edition, Springer Verlag Berlin, 1997.

Applications to industry

1902

1999

Bernstein *polynomials (1912)*

Circular arcs *for curves and surfaces*

Conoids

Cartesian description

Take ***parabols*** *instead circular arcs*

50's

Bézier *curves and surfaces*

UNISURF (P. Bézier)

Gordon surfaces

B-spline *curves and surfaces*

Coons' patches

60's **Parametric description** 70's

SKETCHPAD (Sutherland)
AUTOKON (Mehlun)
EUKLID (Engeli)

CAGD Utah Conf. '74
(Barnhill & Riesenfeld)

NURBS *curves and surfaces*