Equivalent sets of formulas for circumscriptions

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Abstract. Circumscription is a way of using classical logic in order to modelize rules with exceptions and implicit knowledge. Formula circumscription is easier to use in order to modelize a given situation. We describe when two sets of formulas give the same result when circumscribed, introducing two kinds of equivalence. For ordinary equivalence, the two sets give the same circumscription, and for the strong equivalence, when completed by any arbitrary set, the two sets give the same circumscription. The strong equivalence corresponds simply to having the same closure for logical "and" and "or". For the ordinary equivalence, there exists also always a greatest set. Our answer to these two equivalence problems for the case of propositional formula circumscription is exhaustive. This gives rise to various notions of formulas positive with respect to a given set of formulas. When starting from ordinary propositional circumscription, things remain simple enough, and we provide a syntactical description of all these equivalent sets, even in the infinite case.

1 Introduction

Circumscription uses classical logic for representing rules with exceptions. It is often better to use the formula version. An important aspect of formula circumscription has almost not been studied: what are exactly the sets of formulas which give rise to the same circumscription. Answering this question should have important consequences on the automatization of circumscription, and on the knowledge representation side. A possible explanation for the lack of studies on the subject is the complexity of the predicate versions of circumscriptions. We answer fully this problem, providing a syntactical description of most of the sets of formulas concerned, in the propositional case, including the infinite case in order to help the future exploration of the predicate case.

Section 2 introduces propositional circumscriptions. Section 3 gives two kinds of equivalence between sets of formulas, and the two associated notions of "positive formulas". Section 4 examines the infinite case. Section 5 shows that when we start from ordinary circumscriptions, things remain simple (even in the infinite case), providing a syntactical description of all the "equivalent sets" concerned.

2 Propositional circumscription

$\mathbf{L}$ being a propositional logic, $V(\mathbf{L})$ is the set of its propositional symbols. As usual, $\mathbf{L}$ denotes also the set of the formulas. We allow empty sets in *partitions* of $V(\mathbf{L})$. $Th(\mathcal{T}) = \{\varphi \in \mathbf{L}/\mathcal{T} \models \varphi\}$, the set of the *theories* is $\mathbf{T} = \{Th(\mathcal{T})/\mathcal{T} \subseteq \mathbf{L}\}$. Formulas in $\mathbf{L}$ are denoted by letters φ, ψ, subsets of $\mathbf{L}$ by $\mathcal{T}, \Phi, \Psi$, and interpretations for $\mathbf{L}$ (identified with the subset of $V(\mathbf{L})$ that they satisfy) by μ, ν. If $V(\mathbf{L}) = \{P, Q, Z\}$ and $\mu = \{P, Z\}$, then $Th(\mu) = Th(P \wedge \neg Q \wedge Z)$. We define the set $\neg\Phi = \{\neg\varphi/\varphi \in \Phi\}$. $V(\varphi)$ denotes the set of the propositional symbols appearing in φ. $\mathbf{M} = \mathcal{P}(V(\mathbf{L}))$ denotes the set of the interpretations. If $\mathbf{M}' \subseteq \mathbf{M}$, we define $Th(\mathbf{M}') = \{\varphi \in \mathbf{L}/\mu \models \varphi$ for any $\mu \in \mathbf{M}'\}$. This ambiguous meaning of $\models$ and Th is usual. $\mathbf{M}(\mathcal{T})$ denotes the set of the models of $\mathcal{T}$ and TC the classical topological closure: if $\mathbf{M}' \subseteq \mathbf{M}$, $TC(\mathbf{M}') = \mathbf{M}(Th(\mathbf{M}'))$. Generally, a *formula* will be identified with its equivalence class: $\varphi = \psi$ iff $\mathbf{M}(\varphi) = \mathbf{M}(\psi)$.

Definitions 2.1 [14] A *preference relation* in $\mathbf{L}$ is a binary relation $\prec$ over $\mathbf{M}$. $\mathbf{M}_\prec(\mathcal{T})$ is the set of the elements μ of $\mathbf{M}(\mathcal{T})$ *minimal for* $\prec$: $\mu \in \mathbf{M}(\mathcal{T})$ and no $\nu \in \mathbf{M}(\mathcal{T})$ is such that $\nu \prec \mu$.
The *preferential entailment* $f = f_\prec$ is defined by
$f_\prec(\mathcal{T}) = Th(\mathbf{M}_\prec(\mathcal{T}))$, i.e. $\mathbf{M}(f_\prec(\mathcal{T})) = TC(\mathbf{M}_\prec(\mathcal{T}))$. □

Definition 2.2 $\prec$ is *safely founded* **(sf)** if, for any $\mu \in \mathbf{M}(\mathcal{T}) - \mathbf{M}_\prec(\mathcal{T})$, there exists $\nu \in \mathbf{M}_\prec(\mathcal{T})$ such that $\nu \prec \mu$. □
(sf) is also called *stoppered* or *smooth* in the literature.

Property 2.3 [folklore]

1. *If $\prec$ is irreflexive and $f_\prec = f_{\prec'}$, then $\prec = \prec'$.*
2. *If $\prec$ is (sf), then $\prec$ is transitive and irreflexive. If $V(\mathbf{L})$ is finite, $\prec$ is (sf) iff it is transitive and irreflexive.* □

Definition 2.4 [6, 11, 12] $(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is a partition of $V(\mathbf{L})$. $\mathbf{P}$ is the set of the *circumscribed propositional symbols*, $\mathbf{Z}$ of the *variable* ones, the remaining ones, in $\mathbf{Q}$, being *fixed*. A *circumscription* is a preferential entailment $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = f_{\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}}$ where $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$ is defined by:
$\mu \prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})} \nu$ if $\mathbf{P} \cap \mu \subset \mathbf{P} \cap \nu$ and $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$. We define also $\mu \preceq_{(\mathbf{P},\mathbf{Q},\mathbf{Z})} \nu$ if $\mathbf{P} \cap \mu \subseteq \mathbf{P} \cap \nu$ and $\mathbf{Q} \cap \mu = \mathbf{Q} \cap \nu$. □

It is generally better to use a more general version, formula circumscription [6, 11]. Here is the propositional version.

Definition 2.5 $\Phi, \mathcal{T}$ are subsets of $\mathbf{L}$. The *formula circumscription* $CIRCF$ of the formulas of Φ, is as follows: We introduce the set $\mathbf{P} = \{P_\varphi\}_{\varphi \in \Phi}$ of new propositional symbols. $CIRCF(\Phi)(\mathcal{T}) = CIRC(\mathbf{P}, \emptyset, V(\mathbf{L}))(\mathcal{T} \cup \{\varphi \Leftrightarrow P_\varphi\}_{\varphi \in \Phi}) \cap \mathbf{L}$. □

Property 2.6 [2] $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})$.
(In $CIRCF$, $\mathbf{P}$ and $\mathbf{Q}$ are sets of formulas.) □

Definitions 2.7 For any μ, we define the set of formulas $\Phi_\mu = \{\varphi \in \Phi \ / \ \mu \models \varphi\} = Th(\mu) \cap \Phi$. We define two binary relations in $\mathbf{M}$: $\mu \preceq_\Phi \nu$ if $\Phi_\mu \subseteq \Phi_\nu$, and $\mu \prec_\Phi$ if $\Phi_\mu \subset \Phi_\nu$. □

Lemma 2.8 1a. $\mu \prec_\Phi \nu$ iff $\mu \preceq_\Phi \nu$ and $\nu \not\preceq_\Phi \mu$.
1b. $\mu \preceq_\Phi \nu$ iff for any $\varphi \in \Phi$, if $\mu \models \varphi$ then $\nu \models \varphi$.

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2. $\{\varphi\}_\mu \subseteq \{\varphi\}$. Thus, for any $\mu, \mu_i, \nu \in \mathbf{M}$, $\varphi \in \mathbf{L}$, we have ($\mu \preceq_{\{\varphi\}} \nu$ or $\nu \preceq_{\{\varphi\}} \mu$) and not ($\mu_1 \prec_{\{\varphi\}} \mu_2$ and $\mu_2 \prec_{\{\varphi\}} \mu_3$).

3a. $\mu \preceq_\Phi \nu$ iff $\mu \preceq_{\{\varphi\}} \nu$ for any $\varphi \in \Phi$.

3b. $\mu \prec_\Phi \nu$ iff $\mu \preceq_{\{\varphi\}} \nu$ for any $\varphi \in \Phi$, and $\mu \prec_{\{\varphi\}} \nu$ for some $\varphi \in \Phi$.

4. $\prec_\Phi$ and $\preceq_\Phi$ are transitive, $\prec_\Phi$ is irreflexive (thus, $\prec_\Phi$ is a *strict order*) while $\preceq_\Phi$ is reflexive (thus, $\preceq_\Phi$ is a *pre-order*). □

This lemma is immediate. Thus, to know the "useful relation" (see property 2.9-1) $\prec_\Phi$, we need more than each $\prec_{\{\varphi\}}$, we must know all the $\preceq_{\{\varphi\}}$'s, a much more precise information.

Property 2.9 [folklore] *1.* $CIRCF(\Phi) = f_{\prec_\Phi}$.
2. $\prec_\Phi$ *(thus also* $\prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$*) is (sf).* □

Point 1 gives an alternative definition of formula circumscription. Notice that the circumscriptions defined here are the usual propositional adaptations of the original predicate calculus versions of [5, 6, 11]. More details, including the *propositional circumscription axioms*, can be found in e.g. [12, 1, 9, 7]. We refer also to these texts for more details and bibliographical references about the notions reminded above.

3 Equivalences between circumscribed sets

We examine when two sets of formulas Φ and Φ' produce the same formula circumscription. From a knowledge representation perspective, two kinds of such "equivalences" are to be considered.

Definition 3.1 Φ and Φ' are *c-equivalent* ($\Phi \equiv_c \Phi'$), if $CIRCF(\Phi) = CIRCF(\Phi')$. Φ and Φ' are *strongly equivalent* ($\Phi \equiv_{sc} \Phi'$), if, for any set Φ'' of formulas, $CIRCF(\Phi \cup \Phi'') = CIRCF(\Phi' \cup \Phi'')$. □

If $\Phi \equiv_{sc} \Phi'$, then $\Phi \equiv_c \Phi'$. The strong version is useful because, when another rule, or another "individual", is added, this corresponds to an addition of formula(s): e.g., if birds (B_i's) generally fly (F_i), a new bird B_k adds a new formula $B_k \wedge \neg F_k$ to be circumscribed. With standard equivalence, we may then loose this equivalence.

Definitions 3.2 The $\wedge$-*closure of* Φ is the set $\Phi^\wedge = \{\bigwedge_{\varphi\in\Psi} \varphi /$ for any finite $\Psi \subseteq \Phi\}$. The $\vee$-*closure* $\Phi^\vee$ is defined similarly. The $\wedge\vee$-*closure of* Φ is the set $\Phi^{\wedge\vee} = (\Phi^\wedge)^\vee = (\Phi^\vee)^\wedge$. $\Phi^\wedge$ (resp. $\Phi^\vee$, or $\Phi^{\wedge\vee}$) is called a set *closed for* $\wedge$ (resp. *for* $\vee$, or *for* $\wedge$ *and* $\vee$). □

We get always $\top \in \Phi^\wedge, \bot \in \Phi^\vee$ (choose $\Psi = \emptyset$). For the equality $(\Phi^\wedge)^\vee = (\Phi^\vee)^\wedge$, use the de Morgan distribution laws, reminding that we assimilate a formula to its equivalence class.

Definitions 3.3 1. φ is *accessible for* $f = f_\prec$ if $\varphi \in f(\mathcal{T}) - \mathcal{T}$ for some theory $\mathcal{T}$. The set of the formulas *inaccessible for* f is $I_f = I_\prec = \mathbf{L} - \bigcup_{\mathcal{T}\in\mathbf{T}}(f(\mathcal{T}) - \mathcal{T}) = \bigcap_{\mathcal{T}\in\mathbf{T}}(\mathbf{L} - (f(\mathcal{T}) - \mathcal{T}))$.

2. The set of the formulas *positive for* $\prec$ is the set $Pos(\prec)$ of the formulas φ such that, if $\mu \models \varphi$ and $\mu \prec \nu$, then $\nu \models \varphi$. If $\prec = \prec_\Phi$ of definition 2.7, we write $Pos_e(\Phi)$ for the set $Pos(\prec_\Phi)$, called the set of the formulas *positive in* Φ*, in the extended acception*. If $\prec = \preceq_\Phi$, we write $Pos_m(\Phi)$ for the set $Pos(\preceq_\Phi)$ of the formulas *positive in* Φ*, in the minimal acception*. □

Inaccessible formulas for circumscriptions are introduced in [8]. We will show here that in the finite case, I_f is the greatest (for $\subseteq$) set Ψ such that $f = CIRCF(\Phi) = CIRCF(\Psi)$ (theorem 3.8-1b below). As we expect for a set of "positive formulas", $Pos(\prec)$ is always closed for $\wedge$ and $\vee$:

Property 3.4 *1. If* $\prec$ *is (sf),* $Pos(\prec) = I_\prec$.
2. For any $\prec$*,* $Pos(\prec)$ *is closed for* $\wedge$ *and* $\vee$.
3. $\Phi \subseteq Pos_m(\Phi) \subseteq Pos_e(\Phi)$.
4. $\Phi^{\wedge\vee} = Pos_m(\Phi)$. □

Proof: 1. [8, Property 4.9]. 2. and 3. Immediate. The two inclusions in 3 may be strict (see a less trivial example in section 5): If $V(\mathbf{L}) \neq \emptyset$ and $\Phi = \emptyset$, $\preceq_\Phi$ is always satisfied and $\prec_\Phi$ never, thus $Pos_m(\Phi) = Pos(\preceq_\Phi) = \{\top, \bot\}$ and $Pos_e(\Phi) = Pos(\prec_\Phi) = \mathbf{L}$.

4. $\Phi \subseteq Pos_m(\Phi)$ from 3 and $Pos_m(\Phi)$ is closed for $\wedge$ and $\vee$ from 2, thus $\Phi^{\wedge\vee} \subseteq Pos_m(\Phi)$.

Let us suppose now $\varphi \in Pos_m(\Phi) = Pos(\preceq_\Phi)$. As $\{\top, \bot\} \subseteq \Phi^{\wedge\vee}$, we may suppose that there exist μ, ν such that $\mu \models \varphi, \nu \models \neg\varphi$. Then, $\Phi_\mu \not\subseteq \Phi_\nu$ from the definitions of $\preceq_\Phi$ and of $Pos(\preceq_\Phi)$. To any such couple (μ, ν) we associate one formula $\varphi_{(\mu,\nu)} \in \Phi_\mu - \Phi_\nu$. For any ν such that $\nu \models \neg\varphi$, $\{\mathbf{M}(\varphi_{(\mu,\nu)})\}_{\mu\models\varphi}$ is an open cover of $\mathbf{M}(\varphi)$, closed thus compact: there is a finite subcover. To any such ν, we associate φ_ν, the disjunction of the all formulas $\varphi_{(\mu,\nu)}$ involved in some chosen finite cover. As each $\varphi_{(\mu,\nu)}$ is in Φ, $\varphi_\nu \in \Phi^\vee$. Also $\mathbf{M}(\varphi) \subseteq \mathbf{M}(\varphi_\nu)$ and $\nu \notin \mathbf{M}(\varphi_\nu)$, i.e. $\varphi \models \varphi_\nu$ and $\nu \models \neg\varphi_\nu$.

$\{\mathbf{M}(\neg\varphi_\nu)\}_{\nu\models\neg\varphi}$ is an open cover of $\mathbf{M}(\neg\varphi)$ from which we extract a finite subcover to which corresponds a disjunction ψ of formulas $\neg\varphi_\nu$. As each φ_ν is in $\Phi^\vee$, $\psi \in (\neg(\Phi^\vee))^\vee = \neg((\Phi^\vee)^\wedge) = \neg(\Phi^{\wedge\vee})$. Also $\mathbf{M}(\neg\varphi) \subseteq \mathbf{M}(\psi)$, i.e. $\neg\varphi \models \psi$. ψ is a disjunction of $\neg\varphi_\nu$'s which all satisfy $\varphi \models \varphi_\nu$, i.e. $\neg\varphi_\nu \models \neg\varphi$, thus $\psi \models \neg\varphi$. Thus $\varphi = \neg\psi$: $\varphi \in \Phi^{\wedge\vee}$. □

Let us justify the name "positive formulas". It is natural to call the formulas in $\Phi^{\wedge\vee}$, *positive in* Φ, thus our notation $Pos_m(\Phi)$. We think that there are also good reasons to call the formulas in the generally greater set $Pos_e(\Phi)$, *positive in* Φ, in an *extended acception* (see definition 3.3-2, property 3.4, and also the following example and theorem 5.2 below). When considering propositional circumscription $CIRC$, we can be more precise. Let $(\mathbf{P}, \mathbf{Z})$ be a partition of $V(\mathbf{L})$. $(\mathbf{P} \cup \mathbf{Z} \cup \neg\mathbf{Z})^{\wedge\vee}$ is the set of the formulas *positive in* $\mathbf{P}$ *(traditional meaning)*. Let us already detail this important particular case (proofs given below).

Example 3.5 $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi) = f_\prec$, Φ being the set of formulas $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$, i.e. $\prec = \prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})} = \prec_\Phi$.

We need also the relation $\preceq = \preceq_{(\mathbf{P},\mathbf{Q},\mathbf{Z})} = \preceq_\Phi$.

1. $\Phi^{\wedge\vee} = Pos(\preceq) = Pos_m(\Phi)$ is the set of the formulas positive in $\mathbf{P}$, in the traditional meaning, and without element of $\mathbf{Z}$.

2. If $\mathbf{Z} = \emptyset$ or if $\mathbf{P}$ is infinite, then $Pos(\prec) = Pos_m(\Phi) = Pos_e(\Phi)$. Otherwise, the set $Pos(\prec)$ is more complicated. See theorem 5.2-1c and -2 below for the proofs and more details. □

It is convenient to establish now two easy lemmas

Lemma 3.6 If $\Phi \subseteq \Psi \subseteq \Phi^{\wedge\vee}$, we have $\preceq_\Phi = \preceq_\Psi$, thus a fortiori $\prec_\Phi = \prec_\Psi$, i.e. $CIRCF(\Phi) = CIRCF(\Psi)$. □

Proof: We get $\mu \preceq_{\Phi^{\wedge\vee}} \nu$ if $\mu \preceq_\Phi \nu$ (lemma 2.8-1b). From lemma 2.8-3a, $\mu \preceq_\Phi \nu$ if ($\Phi \subseteq \Psi$ and $\mu \preceq_\Psi \nu$), thus $\mu \preceq_\Phi \nu$ if $\mu \preceq_{\Phi^{\wedge\vee}} \nu$. Thus $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$. Thus $\preceq_\Phi = \preceq_\Psi$ iff $\preceq_{\Phi^{\wedge\vee}} = \preceq_{\Psi^{\wedge\vee}}$. Now, $\Phi^{\wedge\vee} = \Psi^{\wedge\vee}$ if $\Phi \subseteq \Psi \subseteq \Phi^{\wedge\vee}$. □

Lemma 3.7 If $V(\mathbf{L})$ is finite, $CIRCF(\Phi) = CIRCF(I_{\prec_\Phi}) = CIRCF(Pos_e(\Phi))$. □

This lemma is contained in [8, Property 5.6] and we refer to [9, Lemma 5.32] for the (easy) complete proof (one of these equalities has also independently appeared as [3, Observation 15]). This result is false in the infinite case (theorem 4.9 below), however it extends to any $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ (theorems 4.9 and 5.2-3a below).

Theorem 3.8 *1a.* $\Phi \equiv_c \Psi$ *iff* $\prec_\Phi = \prec_\Psi$.
If $\Phi \equiv_c \Psi$ *, then* $Pos_e(\Phi) = Pos_e(\Psi)$.
1b. Let us suppose that $V(\mathbf{L})$ *is finite here.*
$\Phi \equiv_c \Psi$ *iff* $\prec_\Phi = \prec_\Psi$ *iff* $Pos_e(\Phi) = Pos_e(\Psi)$.
$\prec_\Phi = \prec_{Pos_e(\Phi)} = \prec_{Pos_m(\Phi)}$.
$Pos_e(\Phi)$ *is the greatest (for* $\subseteq$*) set* Ψ *satisfying* $\Psi \equiv_c \Phi$.

2a. $\Phi \equiv_{sc} \Psi$ *iff* $\preceq_\Phi = \preceq_\Psi$ *iff* $Pos_m(\Phi) = Pos_m(\Psi)$.
Also $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$*, thus* $\prec_\Phi = \prec_{\Phi^{\wedge\vee}}$.
2b. $Pos_m(\Phi) = \Phi^{\wedge\vee}$ *is the greatest (for* $\subseteq$*) set* Ψ *satisfying* $\Psi \equiv_{sc} \Phi$ *(cf lemma 3.6).*

3. $\Phi \cup \{\varphi\} \equiv_c \Phi$ *iff* $\Phi \cup \{\varphi\} \equiv_{sc} \Phi$ *iff* $\varphi \in \Phi^{\wedge\vee}$. □

1 provides a necessary (and sufficient in the finite case) condition for two sets of formulas to give the same circumscription.

2 provides in any case necessary and sufficient conditions for two sets of formulas to be strongly equivalent (meaning to have the same behavior for what concerns circumscription, even when they are completed by new formulas). One of these conditions is very simple: having the same $\wedge\vee$-closure.

The problem of the greatest set c-equivalent to a given set in the infinite case is harder (see theorem 4.9 below). For the smallest sets c-equivalent, or strongly equivalent, to a given set, see [10] (in the finite case only).

3 shows that if we add the formulas one by one, there is identity between c-equivalence and strong equivalence: informally, this comes from the fact that when there is c-equivalence and not strong equivalence between one set and one of its super sets, it is necessary that the added formulas "oppose each other" (this is a consequence of lemma 2.8-3 and of point 2, for examples see theorem 5.2 and example 5.3 below). Such a mutual cancellation is clearly impossible when the two sets differ by only one formula.

Proof: 1a. "iff": Properties 2.3 and 2.9-2. If $\prec_\Phi = \prec_\Psi$, $Pos(\prec_\Phi) = Pos(\prec_\Psi)$, i.e. $Pos_e(\Phi) = Pos_e(\Psi)$ (converse false, see theorem 4.9).

1b. First "iff": 1a. Second "iff": If $\prec_\Phi = \prec_\Psi$, then $Pos(\prec_\Phi) = Pos(\prec_\Psi)$, i.e., $Pos_e(\Phi) = Pos_e(\Psi)$. We suppose now $Pos_e(\Phi) = Pos_e(\Psi)$, then from lemma 3.7 we get $\prec_\Phi = \prec_\Psi$. Maximality of $Pos_e(\Phi)$ comes from $\Phi \subseteq Pos_e(\Phi)$, thus, if $\Psi \equiv_c \Phi$, as $Pos_e(\Psi) = Pos_e(\Phi)$ from 1a, we get $\Psi \subseteq Pos_e(\Psi) = Pos_e(\Phi)$.

In the infinite case, we still get $\prec_\Phi = \prec_{Pos_m(\Phi)} = \prec_{Pos_e(\Phi)}$ for $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$ (theorem 5.2-3a below), but $\prec_\Phi = \prec_{Pos_e(\Phi)}$ can be false for some sets Φ (example 4.2 below).

2a. First "iff". Part "if": From lemma 2.8-3a, if $\preceq_\Phi = \preceq_\Psi$, then for any Ψ', $\preceq_{\Phi\cup\Psi'} = \preceq_{\Psi\cup\Psi'}$.

Part "only if": We suppose $\preceq_\Phi \neq \preceq_\Psi$. Clearly, if $\prec_\Phi \neq \prec_\Psi$, then $\Phi \not\equiv_c \Psi$ thus $\Phi \not\equiv_{sc} \Psi$. Let us suppose $\prec_\Phi = \prec_\Psi$ and $\preceq_\Phi \neq \preceq_\Psi$. There exist μ and ν such that e.g. $\mu \preceq_\Phi \nu$, $\nu \preceq_\Phi \mu$, and $\mu \not\preceq_\Psi \nu$, $\nu \not\preceq_\Psi \mu$. Since $\mu \neq \nu$, there is a formula φ such that $\mu \models \varphi, \nu \not\models \varphi$. We get, from lemma 2.8-3b, $\nu \prec_{\Phi\cup\{\varphi\}} \mu$. Since $\nu \not\preceq_\Psi \mu$, $\nu \not\prec_{\Psi\cup\{\varphi\}} \mu$. This establishes $\Psi \cup \{\varphi\} \not\equiv_c \Phi \cup \{\varphi\}$, thus $\Psi \not\equiv_{sc} \Phi$.

Second "iff". Part "only if": $Pos(\preceq_\Phi) = Pos(\preceq_\Psi)$ if $\preceq_\Phi = \preceq_\Psi$.

Part "if": It is a consequence of the second sentence, proved below: We suppose $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$. Thus, if $Pos(\preceq_\Phi) = Pos(\preceq_\Psi)$, i.e., if $\Phi^{\wedge\vee} = \Psi^{\wedge\vee}$, we get $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}} = \preceq_{\Psi^{\wedge\vee}} = \preceq_\Psi$.

Second sentence: From lemma 3.6 and its proof, we know $\preceq_\Phi = \preceq_{\Phi^{\wedge\vee}}$, thus $\prec_\Phi = \prec_{\Phi^{\wedge\vee}}$.

2b. $Pos_m(\Phi) = \Phi^{\wedge\vee}$ from property 3.4-4. If $\Psi \equiv_{sc} \Phi$, $\Psi^{\wedge\vee} = \Phi^{\wedge\vee}$ from 2a. As $\Psi \subseteq \Psi^{\wedge\vee}$, we get $\Psi \subseteq \Phi^{\wedge\vee}$: $\Phi^{\wedge\vee}$ is maximal.

3. First "iff": Let us suppose $\Phi \cup \{\varphi\} \equiv_c \Phi$, i.e., $\prec_{\Phi\cup\{\varphi\}} = \prec_\Phi$, i.e., from lemma 2.8-3a, ($\mu \preceq_{\{\varphi\}} \nu$ whenever $\mu \prec_\Phi \nu$), and also $\preceq_{\Phi\cup\{\varphi\}} \neq \preceq_\Phi$. Then, from lemma 2.8-3b, there exist μ, ν such that $\mu \preceq_\Phi \nu$, $\nu \preceq_\Phi \mu$, $\mu \not\preceq_{\{\varphi\}} \nu$, $\nu \not\preceq_{\{\varphi\}} \mu$, which contradicts lemma 2.8-2. This means that if $\Phi \cup \{\varphi\} \equiv_c \Phi$, then we must have $\preceq_{\Phi\cup\{\varphi\}} = \preceq_\Phi$, thus $\Phi \cup \{\varphi\} \equiv_{sc} \Phi$.

Second "iff": As $\varphi \in \Phi^{\wedge\vee}$ iff $(\Phi \cup \{\varphi\})^{\wedge\vee} = \Phi^{\wedge\vee}$, 2a above gives the result. □

4 The infinite case: a third set is needed

As we need now a characterization result of formula circumscription, which goes outside our purpose, we list only the main results (see [9] for details and proofs), describing mainly their consequences on the roles of the "positive formulas". Notice that [1, Theorem 7] characterizes $CIRCF$ in the finite case, however [1, Theorem 8] does not help for finding the set replacing $Pos_e(\Phi)$ in the infinite case.

Definitions 4.1 $M_\prec(\mu) = \{\nu \ / \ \mu \prec \nu\}$ and $m_\prec(\mu) = \{\nu \ / \ \nu \prec \mu\}$. We define the equivalence relation $\mu \equiv_\prec \nu$ if $M_\prec(\mu) = M_\prec(\nu)$ and $m_\prec(\mu) = m_\prec(\nu)$. We write $\equiv_\Phi$ for $\equiv_{\prec_\Phi}$. □

We cannot always take $Pos(\prec) = I_\prec$ as our set Φ (cf theorem 3.8-1) (thus, we must find another set given in definition 4.6 below):

Example 4.2 $V(\mathbf{L}) = \{P_i\}_{i\in\mathbb{N}}$. $\nu_i = \{P_0, P_1, \cdots, P_i\}$ ($i \in \mathbb{N}$), $\nu = V(\mathbf{L})$, $\mu = \{P_1\}$. We define the preference relation $\prec$ by $\nu \prec \nu_n$ and $\mu \prec \nu_n$ for any $n \in \mathbb{N}$, and nothing else. $\lim_{i\to\infty} \nu_i = \nu$. We get then $f_\prec = CIRCF(\Phi)$ with $\Phi = \{\varphi \in Pos(\prec) \ / \ \mu \models \varphi$ iff $\nu \models \varphi\}$.

We have here $M_\prec(\mu) = M_\prec(\nu) = \{\nu_i\}_{i\in\mathbb{N}}$ and $m_\prec(\mu) = m_\prec(\nu) = \emptyset$, thus $\mu \equiv_\prec \nu$.

If $\varphi \in (Pos(\prec))_\mu$, then $M_\prec(\mu) = M_\prec(\nu) \subseteq \mathbf{M}(\varphi)$ thus $\nu \in \mathbf{M}(\varphi)$. Thus $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$. As we have $P_0 \in Pos(\prec)$, $\nu \models P_0$ and $\mu \not\models P_0$, we get $(Pos(\prec))_\nu \subset (Pos(\prec))_\mu$ and $\mu \prec_{Pos(\prec)} \nu$. This shows $\prec \neq \prec_{Pos(\prec)}$, i.e., $CIRCF(Pos(\prec)) \neq f_\prec = CIRCF(\Phi)$. □

Definition 4.3 We define $\mu \,\overline{\prec}\, \nu$ if for any φ, ψ such that $\mu \models \varphi$ and $\nu \models \psi$, there exist $\mu' \in \mathbf{M}(\varphi)$, $\nu' \in \mathbf{M}(\psi)$ such that $\mu' \prec \nu'$. □

Remarks 4.4 1. If $\mu \prec \nu$, then $\mu \,\overline{\prec}\, \nu$.

2. If $V(\mathbf{L})$ is enumerable, then $\mu \,\overline{\prec}\, \nu$ iff there exist two sequences with $\lim_{i\to\infty} \mu_i = \mu$, $\lim_{i\to\infty} \nu_i = \nu$ and $\mu_i \prec \nu_i$ for any i.

3. If $V(\mathbf{L})$ is finite, we have $\overline{\prec} \; = \prec$. □

Property 4.5 *1. If* $\mu \,\overline{\prec}\, \nu$ *then* $(Pos(\prec))_\mu \subseteq (Pos(\prec))_\nu$.
2. If $\mu \,\overline{\prec_\Phi}\, \nu$ *then* $\mu \prec_\Phi \nu$ *or* $\mu \equiv_\Phi (\nu)$. □

Definition 4.6 The set of the formulas *positive for a preference relation* $\prec$, in the *restricted acception*, is: $Pos_r(\prec) = \{\varphi \in Pos(\prec)/$ for any μ, ν, if $\mu \,\overline{\prec}\, \nu$, $\mu \not\prec \nu$ and $\nu \models \varphi$, then $\mu \models \varphi\}$.

We write $Pos_r(\Phi)$ for $Pos_r(\prec_\Phi)$. □

Property 4.7 $Pos_r(\prec) \subseteq Pos(\prec)$. *Moreover, if* $\mu \,\overline{\prec}\, \nu$ *then* $(Pos_r(\prec))_\mu \subseteq (Pos_r(\prec))_\nu$. □

Here is a last interesting preliminary result.

Property 4.8 (proof easy) *1.* $\Phi \subseteq Pos_r(\Phi)$.
$Pos_r(\Phi)$ *is stable for* $\wedge$ *and* $\vee$*. Thus:*
$Pos_m(\Phi) \subseteq Pos_r(\Phi) \subseteq Pos_e(\Phi)$.
2. If $V(\mathbf{L})$ *is finite, then* $Pos_r(\prec) = Pos(\prec)$,
thus $Pos_r(\Phi) = Pos_e(\Phi)$. □

Theorem 4.9 ([9, Proposition 6.16], we hope that we have given enough hints here in order to make this result plausible.)

$f_\prec$ is a formula circumscription iff $f_\prec = CIRCF(Pos_r(\prec))$.

Moreover, in this case, $Pos_r(\prec)$ is the greatest set (for $\subseteq$) Φ such that $CIRCF(\Phi) = f_\prec$.

$CIRCF(\Phi) = CIRCF(\Psi)$ iff $Pos_r(\Phi) = Pos_r(\Psi)$.

We can have $Pos_e(\Phi) = Pos_e(\Psi)$ and $\Phi \not\equiv_c \Psi$. □

As we clearly always have $Pos_e(Pos_e(\Phi)) = Pos_e(\Phi)$, the last line is proved by example 4.2 where $Pos_e(\Phi) = Pos(\prec)$.

Example 4.10 $V(\mathbf{L})$ is infinite, μ_1 and μ_2 are distinct interpretations, $\mu \prec \nu$ iff $\mu = \mu_1$ and $\nu = \mu_2$.

$\overline{\prec} = \prec$ (immediate), thus $Pos(\prec) = Pos_r(\prec) = I_\prec$ and $f_\prec = CIRCF(Pos(\prec))$: $f_\prec$ is a formula circumscription, and even an easy one, since $Pos(\prec) = Pos_r(\prec)$. It is easy to check that this is an example of a circumscription falsifying *reverse monotony*: $f_\prec(\mathcal{T}) \cup \mathcal{T}'' \not\models f_\prec(\mathcal{T} \cup \mathcal{T}'')$ (see [9, Example 6.24] for details). This example illustrates the power of theorem 4.9, which detects immediately that this is indeed a formula circumscription. Notice that this natural example of falsification of reverse monotony has already appeared in the literature (e.g. in [13, Example 2.2 (1)]), without noticing that this is a formula circumscription. □

5 A syntactical description, for $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$

For the case of $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$, we describe syntactically all the sets of "positive formulas".

Notations 5.1 $\mathbf{Y}$ is some finite consistent set of literals from $\mathbf{Y}' \subseteq V(\mathbf{L})$. We define the formulas $\bigvee(\mathbf{Y}) = \bigvee_{\varphi \in \mathbf{Y}} \varphi$ and $\bigwedge(\mathbf{Y}) = \bigwedge_{\varphi \in \mathbf{Y}} \varphi$. $\bigvee(\emptyset) = \bot$, $\bigwedge(\emptyset) = \top$. If, for any $Y \in \mathbf{Y}'$, $\mathbf{Y}$ contains Y or $\neg Y$, $\mathbf{Y}$ is *complete* in $\mathbf{Y}'$. □

Theorem 5.2 The case of $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$:

$\prec, \preceq$ and $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$ are as in example 3.5.

We get $Pos(\preceq) \subseteq Pos(\prec) = I_\prec$ (property 3.4).

Also, as $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi)$,

$Pos(\prec) = Pos_e(\Phi)$ and $Pos(\preceq) = Pos_m(\Phi) = \Phi^{\wedge\vee}$

(see example 3.5 and property 3.4-4).

1a. If $\varphi \in Pos_e(\Phi)$, then φ is positive in $\mathbf{P}$ (traditional meaning).

1b. $Pos_m(\Phi) = \Phi^{\wedge\vee} \subseteq Pos_e(\Phi)$.

1c. If $\mathbf{P}$ is infinite or $\mathbf{Z} = \emptyset$, then $Pos_e(\Phi) = Pos_m(\Phi) = \Phi^{\wedge\vee}$.

2. If $\mathbf{P}$ is finite, $Pos_e(\Phi)$ is the set of the disjunctions of formulas of the kind $\bigwedge(\mathbf{P}_a) \wedge \bigwedge(\mathbf{Q}_l) \wedge (\bigwedge(\mathbf{Z}_l) \vee \bigvee(\mathbf{P} - \mathbf{P}_a))$, for $\mathbf{P}_a \subseteq \mathbf{P}$, and for finite sets $\mathbf{Q}_l$ and $\mathbf{Z}_l$ made of literals of $\mathbf{Q}$ and of $\mathbf{Z}$ respectively. Alternatively, we can describe $Pos_e(\Phi)$ as the set of the conjunctions of formulas $\bigvee(\mathbf{P}_a) \vee \bigvee(\mathbf{Q}_l) \vee (\bigvee(\mathbf{Z}_l) \wedge \bigwedge(\mathbf{P} - \mathbf{P}_a))$. If $V(\mathbf{L})$ is finite, we need only to consider the sets $\mathbf{Q}_l$ complete in $\mathbf{Q}$ and $\mathbf{Z}_l$ complete in $\mathbf{Z}$ in these descriptions.

3a. $Pos_r(\Phi) = Pos_e(\Phi)$. Thus $\prec = \prec_\Phi = \prec_{Pos_m(\Phi)} = \prec_{Pos_e(\Phi)}$, i.e. $CIRCF(\Phi) = CIRCF(Pos_m(\Phi)) = CIRCF(Pos_e(\Phi))$.

3b. If $\mathbf{P}$ is infinite or $\mathbf{Z} = \emptyset$, then $Pos_r(\Phi) = Pos_e(\Phi) = Pos_m(\Phi) = \Phi^{\wedge\vee}$. □

Point 1a establishes that any formula "positive in Φ", following our terminology, with $\Phi = \mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q}$, is positive in $\mathbf{P}$, in the traditional meaning. This constitutes part of the justification for our terminology.

Point 1b adapts already given results, to the particular case of ordinary circumscription: The set of the "formulas positive in Φ, for the minimal acception", is the $\wedge\vee$-closure of the set Φ. This set is always included in the set of the "formulas positive in Φ, for the extended acception". This inclusion has allowed us to omit the acception in our comment about 1a, just above.

Point 1c shows that, except when there are variable propositions with a finite number of circumscribed propositions, there is identity between the minimal and the extended acceptions of the "sets of formulas positive in Φ". This is a particularly interesting result. Indeed, it provides an important property (identity between c-equivalence and strong equivalence) which is always true for circumscriptions without variable, and not for the circumscriptions with variables. Moreover, this provides a simple syntactical description of the set of the formulas positive in Φ, even for the extended acception. When applicable ($\mathbf{Z} = \emptyset$ or $\mathbf{P}$ infinite), we get, for any $\Psi \subseteq \mathbf{L}$:

$$CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi) \text{ iff } \Psi^{\wedge\vee} = \Phi^{\wedge\vee}.$$

Point 2 completes point 1c by providing a syntactical description (and even two descriptions) of the set of the formulas positive in Φ, for the extended acception, in the cases not covered by point 1c. Notice that the case $\mathbf{Z} = \emptyset$ of point 2 is immediate: the difficulty of this description comes from the eventual occurrences of elements in $\mathbf{Z}$, which can appear only in some well precised places.

Point 3a shows that the complication of the "restricted acception" for the sets of positive formulas, when $V(\mathbf{L})$ is infinite, is useless for ordinary circumscription. Indeed, the "restricted" and "extended" acceptions are always identical in this case, and the complications seen in section 4 are not needed for ordinary circumscription.

Point 3b comes directly from points 1c and 3a.

Proof: 1a. $\varphi \in Pos(\prec) = I_\prec$ and $\varphi_1 \vee \cdots \vee \varphi_n$ is a *reduced disjunctive normal form* of φ: each φ_i is distinct and, if φ' is a conjunction of literals such that $\varphi_i \models \varphi'$, $\varphi_i \neq \varphi'$, then $\varphi' \not\models \varphi$.

We suppose $\neg P$ appears in φ_i, for $P \in \mathbf{P}$. We call φ'_i the conjunction of the other literals of φ_i. μ is a model of φ'_i. If $\mu \not\models P$, then $\mu \models \varphi_i$ thus $\mu \models \varphi$. Otherwise, with $\nu = \mu - \{P\}$, we have $\nu \prec \mu$ and $\nu \models \varphi_i$ thus $\nu \models \varphi$ and, as $\varphi \in Pos(\prec)$, again $\mu \models \varphi$. Thus $\mathbf{M}(\varphi'_i) \subseteq \mathbf{M}(\varphi)$, a contradiction with the reduced form.

1b. Property 3.4-3 and -4.

1c. If $\mathbf{Z} = \emptyset$, $V(\varphi) \subseteq \mathbf{P} \cup \mathbf{Q}$: cf 1a and 1b.

$\mathbf{P}$ infinite and $\mathbf{Z} \neq \emptyset$. From 1a and 1b we know that if $V(\varphi) \subseteq \mathbf{P} \cup \mathbf{Q}$, then $\varphi \in Pos(\prec)$ iff φ is positive in $\mathbf{P}$. We suppose $V(\varphi) \not\subseteq \mathbf{P} \cup \mathbf{Q}$, and $\varphi_1 \vee \cdots \vee \varphi_n$ is a reduced normal disjunctive form of $\varphi \in Pos(\prec)$. Z is an element in $\mathbf{Z} \cap V(\varphi_i)$. φ'_i is the conjunction of the literals of φ_i without element of $\mathbf{Z}$. Let P be in $\mathbf{P} - V(\varphi)$. Let μ be a model of φ'_i, then there is a model μ' of φ_i such that $\mu' \cap (\mathbf{P} \cup \mathbf{Q}) = \mu \cap (\mathbf{P} \cup \mathbf{Q})$. We define $\nu = \mu \cup \{P\}$. Then, $\mu' \prec \nu$ thus $\nu \models \varphi$. $P \notin V(\varphi)$ and $(\mu \cup \nu) - (\mu \cap \nu) = \{P\}$, thus $\mu \models \varphi$. Thus, $\mathbf{M}(\varphi'_i) \subseteq \mathbf{M}(\varphi)$, which contradicts the reduced form of φ.

2. See [8] or [9, *Proposition 6.32-2*] (not enough room here).

3a. From property 4.7, it suffices to prove $Pos(\prec) \subseteq Pos_r(\prec)$.

Let φ be in $Pos(\prec)$, and two interpretations be such that $\mu \overline{\prec} \nu$, $\mu \not\prec \nu$, and $\nu \models \varphi$. As $(\Phi) \subseteq Pos(\prec)$, from property 4.5-1 we get $(\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})_\mu \subseteq (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})_\nu$: $\mu \cap \mathbf{P} \subseteq \nu \cap \mathbf{P}$ and $\mu \cap \mathbf{Q} = \nu \cap \mathbf{Q}$. As $\mu \not\prec \nu$, we get $\mu \cap \mathbf{P} = \nu \cap \mathbf{P}$. We split the proof in two cases.

Case 1: $\mathbf{P}$ finite. For $\psi = \bigwedge(\mu \cap \mathbf{P}) \wedge \bigwedge(\neg(\mathbf{P} - \mu))$, $\mu \in \mathbf{M}(\psi)$, $\nu \in \mathbf{M}(\psi)$ and, as $\mu \overline{\prec} \nu$, there exist μ', ν' in $\mathbf{M}(\psi)$ with $\mu' \prec \nu'$. Thus, $\mu' \cap \mathbf{P} \subset \nu' \cap \mathbf{P}$, a contradiction with $\mu' \models \psi$ and $\nu' \models \psi$ which forces $\mu' \cap \mathbf{P} = \nu' \cap \mathbf{P}$. Thus we cannot have our hypothesis if $\mathbf{P}$ is finite.

Case 2: $\mathbf{P}$ is infinite. From 1c we get $Pos(\prec) = Pos(\preceq) = (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$, thus $\mu \models \varphi$.

Cases 1 and 2: Thus, from definition 4.6, $\varphi \in Pos_r(\prec)$. This proves $Pos(\prec) = Pos_r(\prec)$.

We get $\prec = \prec_\Phi = \prec_{Pos_m(\Phi)} = \prec_{Pos_e(\Phi)}$. Indeed, we know $\prec = \prec_\Phi = \prec_{Pos_m(\Phi)}$ from property 3.4-4 and theorem 3.8-2a and $\prec = \prec_{Pos_r(\prec)}$ from theorem 4.9.

3b. If P is infinite or $\mathbf{Z} = \emptyset$, we get $Pos(\prec) = Pos(\preceq) = (\mathbf{P} \cup \mathbf{Q} \cup \neg\mathbf{Q})^{\wedge\vee}$ from 1c. We get $Pos_r(\prec) = Pos(\prec)$ from 3a. □

Example 5.3 $\Phi_1 = \mathbf{P} = \{P_1, P_2\}, \Phi_2 = \{P_1 \wedge P_2, P_1 \vee P_2\}$, $(\mathbf{P}, \mathbf{Q}, \mathbf{Z})$ is a partition of $V(\mathbf{L})$ and $\prec = \prec_{(\mathbf{P},\mathbf{Q},\mathbf{Z})}$.

If $\mathbf{Q} = \emptyset$, we have $f_\prec = CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Phi_1)$ from property 2.6. In any case we have $\prec_{\Phi_1} = \prec_{\Phi_2}$, i.e. $CIRCF(\Phi_1) = CIRCF(\Phi_2)$ (lemma 2.8-3b). Thus, $\Phi_1 \equiv_c \Phi_2$.

However, notice that $\Phi_1 \not\equiv_{sc} \Phi_2$. Here are two proofs of this fact:
1) Directly: $\Phi_1 \cup \{P_1\} = \Phi_1$, $\Phi_2 \cup \{P_1\} = \Phi'_2$.
$CIRCF(\Phi_1)(P_1 \vee P_2) = Th(P_1 \Leftrightarrow \neg P_2)$ while
$CIRCF(\Phi'_2)(P_1 \vee P_2) = Th(\neg P_1 \wedge P_2)$: indeed from lemma 2.8-3b we get $\{P_2\} \not\prec_{\Phi_1} \{P_1\}$ and $\{P_2\} \prec_{\Phi'_2} \{P_1\}$.
2) From theorem 3.8-2a: $Pos_m(\Phi_1) = {\Phi_1}^{\wedge\vee} = \{\bot, P_1 \wedge P_2, P_1, P_2, P_1 \vee P_2, \top\}$ while $Pos_m(\Phi_2) = {\Phi_2}^{\wedge\vee} = \{\bot, P_1 \wedge P_2, P_1 \vee P_2, \top\}$.

To keep things simple, let us consider only one variable proposition ($\mathbf{Z} = \{Z\}$) and at most one fixed proposition. We examine successively the cases without and with a fixed proposition:

a) (No fixed proposition) $f_\prec = CIRC(\mathbf{P}, \emptyset, Z)$ ($\mathbf{Q} = \emptyset$):
$\Phi_3 = \{P_1 \wedge P_2 \wedge Z, P_1 \wedge P_2 \wedge \neg Z, P_1 \vee (P_2 \wedge Z), P_1 \vee (P_2 \wedge \neg Z), P_2 \vee (P_1 \wedge Z), P_2 \vee (P_1 \wedge \neg Z), P_1 \vee P_2 \vee Z, P_1 \vee P_2 \vee \neg Z\}$.
Then we have $f_\prec = CIRC(\{P_1, P_2\}, \emptyset, Z) = CIRCF(\Phi_3) = CIRCF(\Phi_1) = CIRCF(\Phi_2)$. This can be checked directly (from lemma 2.8-3b) or by using theorem 5.2-2 from which we get $Pos(\prec) = {\Phi_3}^{\wedge} = {\Phi_3}^{\wedge\vee}$. Thus $f_\prec = CIRCF(Pos(\prec)) = CIRCF(\Phi_3)$ from theorem 5.2-3. From theorem 3.8-1b (a particular case of theorem 4.9), the set $\Psi = I_\prec = Pos(\prec) = Pos_r(\prec) = {\Phi_3}^{\wedge}$ is the greatest one such that $f_\prec = CIRF(\Psi)$. In particular, as $CIRCF(\Phi_1) = CIRCF({\Phi_1}^{\wedge\vee})$, we get ${\Phi_1}^{\wedge\vee} \subseteq {\Phi_3}^{\wedge}$ (easy to check directly).
Thus, $Pos_e(\Phi_1) = Pos_e(\Phi_2) = Pos_e(\Phi_3) = {\Phi_3}^{\wedge}$.

b) (One fixed proposition) $f_\prec = CIRC(\mathbf{P}, Q, Z)$ ($\mathbf{Q} = \{Q\}$):
With $\Psi_i = \Phi_i \cup \{Q, \neg Q\}$ ($i \in \{1, 2\}$), we get $f_\prec = CIRC(\mathbf{P},\mathbf{Q},\mathbf{Z}) = CIRCF(\Psi_1) = CIRCF(\Psi_2)$. The set Ψ_3 replacing Φ_3 above is made from all the φ's in Φ_3 duplicated into the pair $\varphi \vee Q, \varphi \vee \neg Q$, and we get: $I_\prec = Pos(\prec) = Pos_e(\Psi_1) = Pos_e(\Psi_2) = Pos_e(\Psi_3) = {\Psi_3}^{\wedge} = {\Psi_3}^{\wedge\vee}$.

There is always a greatest $\Psi = Pos(\prec)$ such that $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi)$. However, this example shows that it does not generally exist a smallest set Ψ' such that $CIRC(\mathbf{P}, \mathbf{Q}, \mathbf{Z}) = CIRCF(\Psi')$: here Φ_1 and Φ_2 (case a) or Ψ_1 and Ψ_2 (case b) are minimal for $\subseteq$. □

6 Conclusion and future work

We have described all the sets of formulas Φ which, when circumscribed, give rise to the same result as a given set Ψ. Also, we have described all the sets Φ which, when completed by any arbitrary set Φ', give rise to the same result as a given set Ψ, when completed by Φ'. Our description is always syntactical in the second case ("strong equivalence"). In the first case ("standard equivalence"), we have given a syntactical description when we start from an ordinary circumscription (i.e. from a set Ψ made of atoms and of pairs of opposite literals). In the general case (infinite set Φ of arbitrary formulas), it does not seem that such an easy syntactical description exists.

These results should help the automatization of circumscription, because once we know all the equivalent sets, we may start from the best one. It remains to determine which is "the best one", but we have given the first necessary step.

These results should also help the real use of circumscriptions. It is clearly good to know when two sets of formulas give the same circumscription, and also when this equivalence is preserved by the addition of any set of formulas. We would like to describe another interest for modeling complex situations, involving various sets of rules. One way to do this is to associate with each rule a set of formulas to be circumscribed. Then, in order to combine two rules, we could try to make some combinations of the two sets, in order to get a third set, associated with the combination of the individual rules. Various kinds of combinations should be designed, in order to consider cases when e.g. some priority is given to one rule. In order to define precisely such combinations of "sets", it is important to know precisely what are the objects ("sets of formulas") that we want to combine. Our notion(s) of equivalence between sets gives (give) the answer(s). What remains to do is to design such combinations in various configurations (rules with various priorities between them).

Another work is to examine the predicate case. We have developed the infinite propositional case as a first small, but not negligible, step towards the full predicate case.

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