Unifying various approaches to constructive defaults

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Abstract

Default logic is one of the most popular non monotonic formalisms. We show how the classical fix-point definitions of default extensions can be split into a first constructive stage, giving some kind of pre-extensions, and a final stability test. The notion of default reasoning obtained from the pre-extensions has a great interest on its own. It is proved to be equivalent to the notion defined by various authors, among them is the well known proposal of Łukaszewicz. Some modal proposals can also be conveniently studied in this way. So, this text provides constructive definitions of various interesting kinds of pre-extensions. Moreover it shows how a number of already made proposals, including some modal ones, are closely related together even if they look very different.

Keywords: Common sense reasoning, Constructive extensions, Default logic, Modal defaults.

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Notice that the present paper is slightly different (and newer) than the IPMU-92 paper [Moi93]. However, in contrast to the IPMU-92 paper, the present paper unfortunately omits the comparison with [Tru91]. A more complete version is available from the author.

1 Introduction

Default logic is one of the most popular non monotonic formalisms. The aim is to express concisely and naturally rules with exceptions. A great amount of literature has been published on the subject since its introduction by Reiter in 1980. Here, we show how the classical fix point definitions may be split into a constructive stage and a final test. Precisely we start from different definitions of extensions in Reiter's default logic which have been used in the literature. While giving two different ways of calculating extensions, [SR91] and [Lév91] give an interesting variant of the original definitions of [Rei80]. [KBH92] and [TT92] give "semi- constructive" definitions of extensions. Also, following [Kon88], there are "modal simulations" of default logic [SS91, Nie91]. [Łuk88] has defined a slightly different notion which appears regularly since, in many proposals, explicitely or not [GCH90, SS91]. Here, we show how all these definitions relate together. Clearly when the final result is Reiter's extensions, we get an "equivalence" in this meaning, but here we look deeper inside the definitions and show what they have in common, even if the formulations are very different. This comparison allows to tell which definition is "the most constructive". However its main interest in our opinion is to unify these proposals which look so different.

The next section introduces our general framework, with definitions of pre-extensions and of the stability test. We show how these definitions unify several proposals already made and also that justified pre-extensions possess a "constructive definition". Section 3 shows how a semi-constructive definition by [KBH92] of classical default extensions may be considered as an interesting application of our general framework. Also, this is very close to the well-known Łukaszewicz's variant of default logic for which we provide a "constructive" definition. In section 4, we relate our pre-extensions with the literature, showing that the notion of default proof given by the justified pre-extensions is exactly the same as in various proposals. Section 5 shows that some modal translations of default logic can conveniently be studied in our framework.

2 Pre-extensions in default logic

(W, Δ) is a default theory in the meaning of [Rei80]. W is a set of first order formulas in an enumerable language $\mathcal{L}$, and Δ is a set of *defaults* $\delta = \frac{a : b}{c}$ where a, b and c are closed formulas in $\mathcal{L}$. For the sake of simplicity we use only one *justification* b, but we could give our definitions with several justifications.

Notations 2.1 $a = Pre(\delta)$, $b = Jus(\delta)$, $c = Con(\delta)$ and for a set Δ we note $Pre(\Delta) = \{Pre(\delta) \ / \ \delta \in \Delta\}$, and the same for $Jus(\Delta)$ and $Con(\Delta)$.

Definition 2.2 [SR91] A set Δ' of defaults is *grounded in* W when for every default $\delta \in \Delta'$, there exists a finite sequence $(\delta_0, \delta_1, ..., \delta_n)$ in Δ' with $\delta_n = \delta$ such that:

$(W \cup Con(\delta_0, \ldots, \delta_{i-1})) \vdash Pre(\delta_i)$ for $0 \leq i \leq n$ (thus $W \vdash Pre(\delta_0)$).

A set E of formulas is *grounded in* (W, Δ) when there exists a subset Δ' of Δ which is grounded in W and such that $E = Th(W \cup Con(\Delta'))$.

Definition 2.3 A set Δ' of defaults is *justified in* W when for every default $\delta \in \Delta'$ we have $\neg Jus(\delta) \notin Th(W \cup Con(\Delta'))$.

A set E of formulas closed for Th is *justified in* (W, Δ) when there exists a subset Δ' of Δ which is justified in W and such that $Pre(\Delta') \subseteq E$ and $E = Th(W \cup Con(\Delta'))$

Definition 2.4 A set E of formulas in $\mathcal{L}$ is a *pre-extension* of a default theory $T = (W, \Delta)$ iff there exists a sequence E_i with $E = \cup E_i$ and such that:

$$\begin{aligned} E_0 &= Th(W) \\ E_{i+1} &= Th(E_i \cup Con(\Delta_i)) \\ &\quad \text{with} \quad \Delta_i = \{\delta \in \Delta \;/\; Pre(\delta) \in E_i, Con(\delta) \in E\}. \end{aligned}$$

A *generator set* of a pre-extension E is any subset Δ' of Δ, grounded in W, such that $E = Th(W \cup Con(\Delta'))$. $\Delta' = \cup \Delta_i$ is a generator set of the pre-extension E.

Property 2.1 A set E of formulas is a pre-extension if and only if there exists a subset Δ' of Δ, grounded in W and such that $E = Th(W \cup Con(\Delta'))$.

Property 2.2 A set E of formulas is a pre-extension of (W, Δ) if and only if E is closed for Th, $W \subseteq E$, and $\forall \phi \in E$,
$\exists \delta_i$ ($\delta_i \in \Delta$ and $Con(\delta_i) \in E$) for $0 \leq i \leq n$ such that
$W \cup Con(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Pre(\delta_i)$ for $(0 \leq i \leq n)$, and
$W \cup Con(\{\delta_0, \ldots, \delta_n\}) \vdash \phi$.

$$\text{Then } \Delta' = \bigcup_{\phi \in E} \{\delta_0, \ldots, \delta_n\} \text{ is a generator set of } E.$$

The proofs of these two propositions are obvious. Our pre-extensions are grounded. Now, we are interested by the pre-extensions which are also justified, and we note that a pre-extension is justified in (W, Δ) if and only if it has a generator set Δ' which is justified in W. We say also that Δ' is a *justified generator set*.

[Lév91] calls Δ' a *regular* generator set (instead of *justified*) and calls *universe* what we call a grounded set of defaults. What we call justified pre-extension is called *E-set* in [GCH90]. Justified pre-extensions are called *approachable belief set* in [Łuk90], but Łukaszewicz gives the definition only for normal defaults; note also that he gives the corresponding definition (under the name *m-approachable BJ-pair*) for any kind of default for his own modified default theory [Łuk88, Łuk90], described below.

Property 2.3 A set E of formulas in $\mathcal{L}$ is a justified pre-extension of (W, Δ) if and only if $E = \cup E_i$ with:

$$\begin{aligned} E_0 &= Th(W) \\ E_{i+1} &= Th(E_i \cup Con(\Delta_i)) \quad \text{where} \\ &\quad \Delta_i = \{\delta \in \Delta \;/\; Pre(\delta) \in E_i, Con(\delta) \in E, \neg Jus(\delta) \notin E\}. \end{aligned}$$

(Proof easy.) This is very close to the original definition of default extensions, here called R-extensions:

Definition 2.5 [Rei80] A set E of formulas in $\mathcal{L}$ is a *R-extension of* (W, Δ) if and only if $E = \cup E_i$ with:

$$\begin{array}{rcl} E_0 & = & Th(W) \\ E_{i+1} & = & Th(E_i \cup Con(\Delta_i)) \text{ where} \\ & & \Delta_i = \{\delta \in \Delta \ / \ Pre(\delta) \in E_i \text{ and } \neg Jus(\delta) \notin E\}. \end{array}$$

A modification of Reiter's definition such as the one appearing in property 2.3 has been proposed in [GCH90].

Every R-extension is a justified pre-extension, the converse is false. The problem of determining when a justified pre-extension is a R-extension has been tackled by [Lév91, SR91, SS91, KBH92, Bes92] among others. Here, we make precise how these texts relate to each other, and also to [Nie91, TT92].

Before that, let us notice that we may weaken the notion of *justified* pre-extension:

Definition 2.6 A *weakly justified* pre-extension E of (W, Δ) is a set of formulas closed for Th, such that $W \subseteq E$ and for every $\phi \in E$, either $W \vdash \phi$
or there exists δ_i, with $\delta_i \in \Delta$ and $Con(\delta_i) \in E$ for $0 \leq i \leq n$, such that
$W \cup Con(\delta_0, \ldots, \delta_{i-1}) \vdash Pre(\delta_i)$ for $(0 \leq i \leq n)$,
$W \cup Con(\delta_0, \ldots, \delta_n) \vdash \phi$, and
$W \cup Con(\delta_0, \ldots, \delta_n) \not\vdash \neg Jus(\delta_i)$ $(0 \leq i \leq n)$.

Any weakly justified pre-extension is a pre-extension (take $\Delta' = \bigcup_{\phi \in E} \{\delta_0, \ldots, \delta_n\}$ as a generator set). Also, every justified pre-extension is weakly justified, but the converse is not guaranteed as shown by the following example, inspired by example 4.3 in [Łuk88]:

Example 2.1 $W = \emptyset$, $\Delta = \{\delta_i = \frac{: q_i}{a \wedge \neg q_1 \wedge \cdots \wedge \neg q_{i-1}}\}_{i=1,2,3,\ldots}$.

$E = Th(a, \neg q_1, \ldots, \neg q_i, \ldots)$ is a weakly justified pre-extension which is not justified.

It is a pre-extension: in the definition, choose $\Delta_0 = \Delta$, thus $E_1 = Th(W \cup \{a, a \wedge \neg q_1, \ldots\}) = E$.

It is a weakly justified pre-extension: $\phi \in E$ if and only if $(a, \neg q_1, \ldots, \neg q_{i-1}) \vdash \phi$ for some $i \geq 0$ (by compactness of first order logic), that is if and only if: $W \cup Con(\{\delta_i\}) \vdash \phi$. That proves the result because we get also $W \cup Con(\{\delta_i\}) \not\vdash \neg q_i$.

But this is not a justified pre-extension: We cannot find any finite generator set (otherwise one δ_i would suffice, with i being the maximum of the indices of the defaults in this generator set, and $\neg q_i$ would not be in E). Moreover, for any infinite subset Δ' of Δ, for any $\delta_j \in \Delta'$, there is at least one $\delta_k \in \Delta'$ such that $Con(\delta_k) \vdash \neg q_j$.

In fact, for any *finite* Δ, there is equivalence between weakly justified and justified pre-extensions. For infinite Δ, it can be argued that the notion of weakly justified pre-extension is the best one.

Note that we have:

Property 2.4 A (weakly) justified pre-extension is unconsistent if and only if W itself is unconsistent.

Indeed, as soon as there exists a non empty generator set $\Delta' \supseteq \{\delta\}$, $E \not\vdash \neg Jus(\delta)$ thus $E \not\vdash \bot$. For a *weakly* justified pre-extension: if $E \vdash \bot$, then $\exists(\delta_i)_{1 \leq i \leq n} W \cup$

$Con(\delta_1, \ldots, \delta_n) \vdash \bot$ and $W \cup Con(\delta_1, \ldots, \delta_n) \not\vdash \neg Jus(\delta_i)$ for $1 \leq i \leq n$, wich implies $n = 0$ ($\{\delta_i\}_{1 \leq i \leq n} = \emptyset$).

A justified pre-extension always exists. Moreover, it is easy to give a *constructive definition* (not using the final result E in the steps used to define the successive E_i). This affirmation may seem paradoxical as the characteristic property 2.3 seems to be "less constructive" than the definition of R-extensions: the final result E appears twice instead of once only. But it is easy to give other equivalent definitions such as the following one:

Property 2.5 E is a justified pre-extension of (W, Δ) iff there exists two sequences $(E_i)_{i \in \mathbb{N}}$, $(F_i)_{i \in \mathbb{N}}$ of sets of formulas such that $E = \cup E_i$ and:
– $E_0 = W$, $F_0 = \emptyset$;
– for $i \geq 0$: if there is some $\delta_i \in \Delta$ such that $Pre(\delta_i) \in E_i$ and for any $\phi \in F_i \cup \{Jus(\delta_i)\}$, we have $E_i \cup Con(\delta_i) \not\vdash \neg\phi$,
then define $E_{i+1} = E_i$ and $F_{i+1} = F_i$
or $E_{i+1} = Th(E_i \cup Con(\delta_i))$ and $F_{i+1} = F_i \cup \{Jus(\delta_i)\}$,
else define $E_{i+1} = E_i$ and $F_{i+1} = F_i$.

The proof is easy. This characterization reminds some definitions appearing in the literature, e. g. [Łuk88, Bes92], even if it is slightly different. Note that this characterization is not only "constructive" but of a "one default at a time" kind. However, there may be an infinite number of steps and there is undeterminism involved at each step. Here is another way of expressing the same property, starting from a given sequence of defaults, to be compared with other definitions of this kind appearing later in this text:

Property 2.6 E is a justified-extension of (W, Δ) iff there exists a sequence $\overline{\Delta'}$ of defaults[1] in Δ ($\overline{\Delta'} = \delta_0, \delta_1, \delta_2, \cdots$) and two sequences (E_i), (F_i) of sets of formulas such that $E = \cup E_i$ and:
- $E_0 = W$, $F_0 = \emptyset$;
- for $i \geq 0$: if $Pre(\delta_i) \in E_i$ and for any $\phi \in F_i \cup \{Jus(\delta_i)\}$, we have $E_i \cup Con(\delta_i) \not\vdash \neg\phi$,
then $E_{i+1} = Th(E_i \cup Con(\delta_i))$ and $F_{i+1} = F_i \cup \{Jus(\delta_i)\}$,
else $E_{i+1} = E_i$ and $F_{i+1} = F_i$.

Now, if we want to obtain R-extensions, we need another property.

Definition 2.7 A set E of formulas satisfying $E = Th(E)$ is *stable for* Δ iff for any default $\delta \in \Delta$: if $Pre(\delta) \in E$ and $\neg Jus(\delta) \notin E$ then $Con(\delta) \in E$.

We call *stability test*, or *st-test*, the operation consisting in checking for stability of a given set E.

This property appears in several texts about default logic, moreover it is an adaptation of a property appearing in the founding paper [Rei80]. It is clear that any R-extension is

[1]We may restrict our attention to sequences which are enumerations of subsets $\Delta' \subseteq \Delta$.

a justified pre-extension which is stable. And now we have given enough properties: the converse is true. As this is a relatively well known result, we do not give the proof here. We have split the definition of R-extensions into three properties: groundedness, justification and stability. Now, if we examine the problem of constructiveness of R-extensions with respect to these three properties, we remark that we need the final result to ensure stability.
– Groundedness is constructive, and even it is much easier to guarantee this condition in the construction of extensions than to give a definition without taking care of groundedness and then to verify this property. Note however that [SR91] gives an effective way of computing R-extensions which firstly does not take care of groundedness, and then checks it. A priori such a method looks very inefficient.
–Justification can easily be made constructive also: it suffices to take care of all the justifications needed at the previous steps and to respect the new justifications at each step.
–Stability is the only inherently non constructive property of these three. The problem comes from the double condition $Pre(\delta) \in E$ and $\neg Jus(\delta) \notin E$ to be checked for every default in Δ. If we construct our pre-extensions "from below" (that is, with growing pre-extensions as in property 2.5), we have problems with the $\neg Jus(\delta) \notin E$ part, and if we design a construction "from above" (decreasing pre-extensions, as in [SR91]), we have problems with the $Pre(\delta) \in E$ part.

There have been several "semi-constructive" definitions of R-extensions in the literature, consisting in a first constructive stage, giving some class of pre-extensions, followed by a final "test". It appears that in a lot of cases, the final test either is a stability test or is a more complicated test including a stability test. So, it is important to compare these various notions of "pre-extensions". We have seen that it is easy to give a constructive definition of justified pre-extensions, but we may try to go further, the constructive stage giving as few potential pre-extensions to test as possible. However, we must be careful in not introducing a notion of pre-extensions which would be too complicated to construct. Note also that if a proposal gives even more pre-extensions than the justified pre-extensions while the kind of pre-extension obtained is harder to construct than the pure justified pre-extensions, its interest is arguable.

3 Two particular kinds of constructive pre-extensions

If we want our definition to be as constructive as possible, we need to give the minimum work to the final stability test, that is we need to try to restrict the class of pre-extensions to be tested. One of the most interesting solutions in this direction has been made by [KBH92]. The interesting part of their definition is that their preliminary constructive stage is simple and natural, and it gives only a rather restricted sort of justified pre-extensions, this sort being sufficient to give all the R-extensions. We call here "K-extensions" the pre-extensions they get.

Definition 3.1 (adaptation of a definition appearing in [KBH92]) E is a *K-extension* iff there exists two sequences (E_α), (F_α) of sets of formulas such that $E = \cup E_\alpha$ and:
- $E_0 = W$, $F_0 = \emptyset$;

- for any ordinal α other than 0, we define $E'_\alpha = \bigcup_{\beta<\alpha} E_\beta$, $F'_\alpha = \bigcup_{\beta<\alpha} F_\beta$;

if there is some $\delta_\alpha \in \Delta$
such that $E'_\alpha \vdash Pre(\delta_\alpha)$,
$E'_\alpha \not\vdash Con(\delta_\alpha)$ and
for any $\phi \in F'_\alpha \cup \{Jus(\delta_\alpha)\}$, we have $E'_\alpha \cup \{Con(\delta_\alpha)\} \not\vdash \neg\phi$,
then choose one of these δ_α and
define $E_\alpha = Th(E'_\alpha \cup \{Con(\delta_\alpha)\})$ and $F_\alpha = F'_\alpha \cup \{Jus(\delta_\alpha)\}$,
else define $E_\alpha = E'_\alpha$ and $F_\alpha = F'_\alpha$.

In this last case we have $E_\alpha = E_\beta$, $F_\alpha = F_\beta$ for any ordinal $\beta \geq \alpha$, thus $E = E_\alpha$.
In any case, if α is not a limit ordinal, we have $E'_\alpha = E_{\alpha-1}$, $F'_\alpha = F_{\alpha-1}$.

The only two real differences with justified pre-extensions (see property 2.5) are that firstly if we can increase E_α here we must do it and secondly there is the additional condition $E_\alpha \not\vdash Con(\delta)$, that is only the defaults which are of some utility are used. Thus K-extensions are justified pre-extensions, the converse being not guaranteed.

An apparent third difference is that here we must consider ordinals greater than ω, in order to be sure that we do not stop too early (see [FM92] for a study of this problem when dealing with constructive definitions of default extensions). This complication is not needed in property 2.5 as anyway with justified pre-extensions we may "stop too early". Clearly we could continue the construction beyond ω there also. It would make no difference at all as in property 2.5 we may decide arbitrarily to pose $E_i = E_{i-1}$, and as anyway if we get an "incomplete pre-extension" there exists always an ordering of the defaults in Δ which would allow to go as far as necessary. Thus this difference in the presentation of the definitions involved could be suppressed (our languages are enumerable).

For those who do not like this notion of "constructivity beyond ω", here is an equivalent definition not using ordinals.

Definition 3.2 E is a *K-extension* iff there exists an enumeration $\overline{\Delta} = \delta_0, \delta_1, \delta_2, \cdots$ of all the defaults in Δ, a sequence $(\overline{\Delta}^i = \delta^i_0, \delta^i_1, \delta^i_2, \cdots)_{i\geq 0}$ of sequences of defaults in Δ and two sequences (E_i), (F_i) of sets of formulas such that $E = \cup E_i$ and:
- $E_0 = W$, $F_0 = \emptyset$, $\overline{\Delta}^0 = \overline{\Delta}$;
- for any integer $i \geq 0$:

if $E_i \vdash Con(\delta^i_0)$
or for some $\phi \in F_i \cup \{Jus(\delta^i_0)\}$, we have $E_i \cup \{Con(\delta^i_0)\} \vdash \neg\phi$
then $E_{i+1} = E_i$, $F_{i+1} = F_i$, and $\overline{\Delta}^{i+1} = \delta^i_1, \delta^i_2, \delta^i_3, \cdots$;
else if $E_i \not\vdash Pre(\delta^i_0)$
then $E_{i+1} = E_i$, $F_{i+1} = F_i$, and $\overline{\Delta}^{i+1} = \delta^i_1, \delta^i_2, \cdots, \delta^i_i, \delta^i_o, \delta^i_{i+1}, \cdots$;
else $E_{i+1} = Th(E_i \cup \{Con(\delta^i_0)\})$, $F_{i+1} = F_i \cup \{Jus(\delta^i_0)\}$, $\overline{\Delta}^{i+1} = \delta^i_1, \delta^i_2, \delta^i_3 \cdots$

Property 3.1 Any default theory has a K-extension.

The proof is obvious, and comes form the "constructive nature" of definition 3.1.

Property 3.2 [KBH92] A K-extension is a R-extension of (W, Δ) if and only if it is stable.

It appears that K-extensions are very close to the m-extensions of [Łuk88, Łuk90] which we call Ł-extensions.

Definition 3.3 [Łuk88] S and U being two sets of formulas, $\Gamma_1(S,U)$ and $\Gamma_2(S,U)$ are the two smallest sets of formulas verifying:
1) $\Gamma_1(S,U) = Th(\Gamma_1(S,U))$;
2) $W \subseteq \Gamma_1(S,U)$;
3) for all $\delta \in \Delta$ such that $Pre(\delta) \in \Gamma_1(S,U)$, and
for all $\phi \in U \cup \{Jus(\delta)\}$, we have $S \cup \{Con(\delta)\} \not\vdash \neg\phi$,
then $Con(\delta) \in \Gamma_1(S,U)$ and $Jus(\delta) \in \Gamma_2(S,U)$.

E is a *Ł-extension* if and only if there exists a F such that $E = \Gamma_1(E,F)$ and $F = \Gamma_2(E,F)$.

From this definition, the analogies with K-extensions are not obvious. But we may characterize Ł-extensions in a way more interesting for us.

Property 3.3 E is a Ł-extension iff there exists a set F, a sequence (Δ_α) of sets of defaults and two sequences (E_α), (F_α) such that $E = \cup E_\alpha$, $F = \cup F_\alpha$ and:
- $E_0 = W$, $F_0 = \emptyset$, $\Delta_0 = \Delta$;
- for any ordinal α other than 0, define $E'_\alpha = \bigcup_{\beta<\alpha} E_\beta$, $F'_\alpha = \bigcup_{\beta<\alpha} F_\beta$, $\Delta'_\alpha = \bigcap_{\beta<\alpha} \Delta_\beta$;

if there is some $\delta_\alpha \in \Delta'_\alpha$
such that $E'_\alpha \vdash Pre(\delta_\alpha)$ and
for any $\phi \in F'_\alpha \cup \{Jus(\delta_\alpha)\}$, we have $E'_\alpha \cup \{Con(\delta_\alpha)\} \not\vdash \neg\phi$,
then choose one of these δ_α and
define $E_\alpha = Th(E'_\alpha \cup \{Con(\delta_\alpha)\})$, $F_\alpha = F'_\alpha \cup \{Jus(\delta_\alpha)\}$, $\Delta_\alpha = \Delta'_\alpha - \{\delta_\alpha\}$,
else define $E_\alpha = E'_\alpha$, $F'_\alpha = F_\alpha$ and $\Delta_\alpha = \Delta'_\alpha$.

This proposition is a slight adaptation of theorem 5.18 in [Łuk88]. In [Łuk88] there is no explicit use of ordinals and thus no obligation of going beyond ω when necessary, but there is an additional Ł-stability test (the corresponding of the stability test for Ł-extensions). In fact this test is a rewriting of the condition appearing in each stage, and as the property is written here, the final test of [Łuk88] is unnecessary. Thus, there exists a "constructive" (and even a "one default at a time") definition of Ł-extensions.

Again, we give an equivalent property using sequences of sequences instead of ordinals beyond ω:

Property 3.4 E is a *Ł-extension* iff there exists an enumeration $\overline{\Delta} = \delta_0, \delta_1, \delta_2, \cdots$ of all the defaults in Δ, a sequence $(\overline{\Delta}^i = \delta^i_0, \delta^i_1, \delta^i_2, \cdots)_{i\geq 0}$ of sequences of defaults in Δ and two sequences (E_i), (F_i) of sets of formulas such that $E = \cup E_i$ and:
- $E_0 = W$, $F_0 = \emptyset$, $\overline{\Delta}^0 = \overline{\Delta}$;
- for any $i \in \mathbb{N}$:
if for some $\phi \in F_i \cup \{Jus(\delta^i_0)\}$, we have $E_i \cup \{Con(\delta^i_0)\} \vdash \neg\phi$
then $E_{i+1} = E_i$, $F_{i+1} = F_i$, and $\overline{\Delta}^{i+1} = \delta^i_1, \delta^i_2, \delta^i_3, \cdots$;
else if $E_i \not\vdash Pre(\delta^i_0)$
then $E_{i+1} = E_i$, $F_{i+1} = F_i$, and $\overline{\Delta}^{i+1} = \delta^i_1, \delta^i_2, \cdots, \delta^i_i, \delta^i_o, \delta^i_{i+1}, \cdots$;
else $E_{i+1} = Th(E_i \cup \{Con(\delta^i_0)\})$, $F_{i+1} = F_i \cup \{Jus(\delta^i_0)\}$, $\overline{\Delta}^{i+1} = \delta^i_1, \delta^i_2, \delta^i_3 \cdots$

The only difference between Ł-extensions and K-extensions is that in definition 3.1 of K-extensions there is the additional requirement $E_i \not\vdash Con(\delta)$ in order to be allowed to use a default δ. Thus any K-extension is a Ł-extension, the converse is not guaranteed as shown by the following example, even if K-extensions and Ł-extensions are rather close together.

Example 3.1 (cf example 4.2 in [Łuk88])
$W = a$, $\Delta = \{\delta_1 = \frac{:c}{a}, \delta_2 = \frac{:b}{\neg c}\}$.

First, we look for K-extensions.
$E_0 = \{a\}, F_0 = \emptyset$,
δ_1 ? no because $E_0 \vdash a$ (this is the difference with Ł-extensions).
δ_2 ? yes because $E_0 \not\vdash \neg c, E_0 \cup \{\neg c\} \not\vdash \neg b$. Thus $E_1 = Th(a, \neg c), F_1 = \{b\}$.
There is exactly one K-extension $E = E_1 = Th(a, \neg c)$. Note that E is also a R-extension.

However there are two Ł-extensions $E' = Th(a)$, associated with $F' = \{c\}$ and
$E = Th(a, \neg c)$, associated with $F = \{b\}$.

From this example, one could conclude that any that K-extension is a maximal Ł-extension. It would be a good thing since the existence of non maximal Ł-extensions is one weakeness of Łukaszewicz's proposal. Although it is true that any maximal Ł-extension is a K-extension, unfortunately the converse is false: as the following example shows, there are some non maximal Ł-extensions which are not eliminated by the additional requirement of definition 3.1.

Example 3.2 $W = \emptyset$, $\Delta = \{\delta_1 = \frac{:j_1}{a}, \delta_2 = \frac{:j_2}{a \wedge \neg j_1}\}$.

There are two K-extensions, which are also Ł-extensions:
$E = Th(a)$, associated with $F = \{j_1\}$, and
$E' = Th(a, \neg j_1)$, associated with $F' = \{j_2\}$.

Note that here st-test eliminates E, E' being the only R-extension.

From property 2.5 it is clear that any Ł-extension is a justified pre-extension.

Also it is easy to show from property 3.3 that any default theory has at least one Ł-extension.

4 Justified pre-extensions in the literature

To continue our comparison with the existing literature about default logic, we study here a proposal made in[GCH90]. We have yet signaled that they too introduce justified pre-extensions, under the name "E-sets". Their goal was to define a "new kind" of default extensions. And in fact they come up with a kind of extensions which is very close to Ł-extensions. As this fact is not mentioned in their text, we precise this point now.

Definition 4.1 (adapted from [GCH90]) Let $\mathcal{T} = (W, \Delta)$ be a default theory. Any set E which is a union of justified pre-extensions of $\mathcal{T}$ and which is closed for Th (i.e. $Th(E) = E$) is a *C-pre-extension* of $\mathcal{T}$. Any C-pre-extension of $\mathcal{T}$ which is maximal for $\subseteq$ is a *C-extension* of $\mathcal{T}$.

As justified pre-extensions are closed for Th, the only unions of justified pre-extensions which are closed for Th are the unions $\cup_{i\in I}E_i$, containing an inclusion chain $E_{j_1} \subseteq E_{j_2} \subseteq \cdots$ where $J = \{j_1, j_2, \ldots\} \subseteq I$, such that for any $i \in I$ there exists a $j \in J$ with $E_i \subseteq E_j$ (thus $\subseteq \cup_{j\in J}E_j$). As afterwards only the maximal C-pre-extensions will be considered, the only reason for which these unions are needed at all is in the case of infinite chains of justified pre-extensions as in example 2.1. In example 2.1, $Th(a) \cup Th(a \wedge \neg j_1) \cup Th(a \wedge \neg j_1 \wedge \neg j_2) \cup \cdots = Th(a, \neg j_1, \neg j_2, \ldots)$ is a C-extension but it is not a Ł-extension. This is the only case where C-extensions are not also Ł-extensions. Anyway, the notion of *"default proof"*, sometimes also called the *"extension membership problem"*, is exactly the same with [Łuk88] and [GCH90]. This means that a given formula ϕ belongs to a Ł-extension if and only if it belongs to a C-extension. To see this, it suffices to compare the definition of a *C-proof* (definition 4 in [GCH90]) and the characterization of *Ł-proofs* as given in theorem 7.7 of [Łuk88] or theorem 5.126 in [Łuk90], they coincide[2]:

Property 4.1 [Łuk88, Łuk90] (resp. [GCH90]) A sentence ϕ belongs to a Ł-extension (resp. a C-extension) of (W, Δ) if and only if there exists a finite sequence $\Delta_0, \cdots, \Delta_k$ of finite subsets of Δ such that:

(1) $W \cup Con(\Delta_0) \vdash \phi$;
(2) $W \cup Con(\Delta_i) \vdash Pre(\Delta_{i-1})$, $(1 \leq i \leq k)$;
(3) $\Delta_k = \emptyset$;
(4) $W \bigcup \bigcup_{i=0}^{k} Con(\Delta_i) \not\vdash \neg\psi$ for any $\psi \in \bigcup_{i=0}^{k} Jus(\Delta_i)$.

Note that this result holds also with K-extensions, with justified pre-extensions, and with weakly justified pre-extensions instead of Ł- (or C-)extensions. Regarding the respective extensions, here are the precise relations; the proofs (omitted) are easy:

Property 4.2 • Any maximal Ł-extension is a C-extension.

• Any C-extension is either a maximal Ł-extension or a union of an infinite chain of Ł-extensions.

• Also, we have: any K-extension is a Ł-extension, any maximal K-extension is a C-extension.

• Any C-extension is a weakly justified pre-extension.

• Remind that any Ł-extension is a justified pre-extension.

We compare now the justified pre-extensions to a recent proposal.

Definition 4.2 [TT92] A set E of formulas in $\mathcal{L}$ is a *σ-constructive extension of* (W, Δ) iff $E = \cup E_i$ with:

$$\begin{aligned} E_0 &= Th(W) \\ E_{i+1} &= Th(E_i \cup Con(\Delta'_i)) \text{ where} \\ &\quad \Delta'_i \subseteq \{\delta \in \Delta \ / \ Pre(\delta) \in E_i \text{ and } \neg Jus(\delta) \notin E_i\}. \end{aligned}$$

The two differences with standard R-extensions (definition 2.5) are:

[2] Note however that this was not readily apparent due to a mistake in theorem 7.7 in [Łuk88]. This mistake was not corrected until [Łuk90]. This may explain why [GCH90] did not notice this identity.

1) As in property 2.5, when we may use one default δ, we have the choice of using it or not: Δ'_i is any subset of the corresponding Δ_i of definition 2.5.

2) The condition involving the justifications is milder, in order to get a "constructive process": it is only checked whether the preceding E_i does not contradict the justification $Jus(\delta)$, but the final results E may happen to contradict it.

Thus, we have a kind of "partially justified pre-extension": it is easy to prove that any justified pre-extension is a σ-constructive extension, and that any σ-constructive extension is a pre-extension, none of the converses being guaranteed.

Here is how [TT92] test whether their pre-extensions are R-extensions or not:

Property 4.3 [TT92] A σ-constructive extension is a R-extension if and only if it is justified and stable[3]. Moreover, any R-extension is a σ-constructive extension.

To compare this semi-constructive definition with the preceding ones, we remark that there are more σ-constructive extensions than justified pre-extensions, moreover the final test is more complicated than the stabiblity test alone. The definition of the σ-constructive extensions is not completely constructive in our meaning because consistency is not guaranteed: it is the final test which eliminates all the inconsistent σ-constructive extensions. Thus the notion of default proof for the σ-constructive pre-extensions, i.e. the σ-constructive pre-extension membership problem, is meaningless. It remains to see whether some constraints on the choice of the defaults selected at each step, as suggested by [TT92], could radically improve the efficiency. Also, Tan[4] has proved that the justified σ-constructive extensions coincide exactly with the justified pre-extensions.

5 Pre-extensions in modal translations of default logic

Now, we examine two seemingly different proposal [SS91, Nie91]and show that they have several profound analogies with our global presention of R-extensions as stable justified pre-extensions. Several authors have "translated" default logic into a modal formalism. [SS91] chooses the modal system T.

Definition 5.1 (reminder) The modal *system* T adds to classical first order logic a modal operator L with the following properties:
– axiom schema of *necessity*: $L\phi \Rightarrow \phi$, for any formula ϕ;
– axiom schema of *consequence*: $L(\phi \Rightarrow \psi) \Rightarrow (L\phi \Rightarrow L\psi)$;
– axiom schema of *Barcan*: $\forall x(L\phi(x)) \Rightarrow L(\forall x\phi(x))$;
– rule of *necessitation*: If $\vdash \phi$ then $\vdash L\phi$ (if ϕ is a tautology, so is $L\phi$).

Also the modal operator M is $\neg L \neg$.

[3][TT92] uses "*closed under* Δ" instead of "stable for Δ" and "*non self-defeating*" instead of "justified".
[4]Personal communication.

Property 5.1 (reminder) The main properties of T are:

– If $\vdash \phi \Leftrightarrow \psi$, then $\vdash L\phi \Leftrightarrow L\psi$	– $L(\phi \wedge \psi) \Leftrightarrow (L\phi \wedge L\psi)$
– $L(\psi \Rightarrow \psi)$	– $\psi \Rightarrow M\psi$ (i.e. $\psi \Rightarrow \neg L\neg\psi$)
– $L\neg\psi \Rightarrow \neg L\psi$	– $L\psi \Rightarrow LM\psi$
– $L\psi \Rightarrow L(\psi \vee \phi)$, thus	– $(L\psi \vee L\phi) \Rightarrow L(\psi \vee \phi)$, but the converse is false.

In terms of Kripke models, system T corresponds to a **reflexive** relation between the" worlds". Also as here we accept to start from full first order logic (with individual variables and predicate symbols) and as the Barcan formula is given as an axiom (which is not always done in "predicative" systems T), all the "worlds" must have the same domain.

The system of [SS91] contains another modal operator H which respects only one property: $H\phi \Rightarrow \neg L\neg\phi$ (for any formula ϕ). Any term $H\phi$ is considered as another propositional variable. Here is how the authors translate a default theory $\mathcal{T} = (W, \Delta)$ (where W is supposed consistent):

Definition 5.2 [SS91] $\mathcal{T} = (W, \Delta)$ gives rise to $\mathcal{T}^M = \{L\phi \;/\; \phi \in W\} \cup$
$\{(La \wedge Hb \Rightarrow Lc) \;/\; \frac{a:b}{c} \in \Delta\} \cup \{(Hb \Rightarrow \neg L\neg b) \;/\; b \in Jus(\Delta)\}$.
A set E^M of modal formulas is a *modal S-extension* of $\mathcal{T}^M$ if and only if $\mathcal{T}^M \subseteq E^M$ and
1) For any default $\delta = \frac{a:b}{c} \in \Delta$, either $Hb \in E^M$ or $\neg Hb \in E^M$.
2) $Th_T(E^M) = E^M$: E^M is deductively closed for the consequence relation of system T.
3) E^M is consistent, and E^M is maximally consistent with respect to the set of all the Hb's possible, i.e. it is not possible to add another Hb to E^M (as in point 1) without provoking an inconsistency.
4) E^M is a minimal set with these properties.

In fact, we are interested mainly by the objective formulas (i.e. without any modal operator) ϕ such that $L\phi \in E^M$. The name *S-extension* will be used here to designate this objective subset E of E^M: $E = \{\phi \;/\; L\phi \in E^M\}$.

Property 5.2 [SS91] A S-extension E is a R-extension if and only if E^M respects the additional condition: *For any $b \in Jus(\Delta)$, if $\neg Hb \in E^M$, then $L\neg b \in E^M$.*

We call this test the *S-test*.

Moreover any R-extension can be obtained in this way.

Let us first note a strange behavior of the S-extensions so defined:

Example 5.1 $W = \emptyset$, $\Delta = \{\delta_1 = \frac{:b}{c}, \delta_2 = \frac{:b}{\neg c}\}$.

There are two K-extensions, which are also the two Ł-extensions:

$E_1 = Th(c)$, associated with $F = \{b\}$ and $E_2 = Th(\neg c)$ associated with the same F.

There is no R-extension.

The translation of [SS91] gives: $\mathcal{T}^M = \{Hb \Rightarrow Lc, Hb \Rightarrow L\neg c\}$. As $\mathcal{T}^M \subseteq E^M$ and $E^M = Th_T(E^M)$, we get $(\neg Lc \vee \neg L\neg c) \Rightarrow \neg Hb \in E^M$. Now, $Lc \Rightarrow c$ and $L\neg c \Rightarrow \neg c$ are in E^M, as any axiom of the system T, thus $\neg Lc \vee \neg L\neg c \in E^M$. This gives: $\neg Hb \in E^M$. Thus, the only modal S-extension E^M of $\mathcal{T}^M$ is $Th_T(\mathcal{T}^M)$ (no default is applicable): $E = Th(\emptyset)$.

Imagine now that we start from:

$W' = \{b_1 \Leftrightarrow b_2\}$, $\Delta' = \{\delta_1 = \frac{:b_1}{c}, \delta_2 = \frac{:b_2}{\neg c}\}$.

Again, as expected, there is no R-extension and we get two Ł-extensions (or K-extensions):

$E_1 = Th(c)$, associated with $F_1 = \{b_1\}$ and $E_2 = Th(\neg c)$ associated with $F_2 = \{b_2\}$.

However, with the S-extensions, we get a different result now.

$\mathcal{T}'^M = \{Hb_1 \Rightarrow Lc, Hb_2 \Rightarrow L\neg c\}$, giving two S-extensions:

– $Hb_1 \in E_1'^M$, then we have $Lc \in E_1'^M$, thus $\neg L\neg c \in E_1'^M$ and $\neg Hb_2 \in E_1'^M$, this giving our first S-extension $E_1' = Th(c)$.

– $Hb_2 \in E_2'^M$. For the same reasons, this gives the second S'-extension $E_2' = Th(\neg c)$.

We consider this behavior as very contestable: these two default theories should have the same extensions. This behavior requests two defaults with the same justification, which is rather exceptional, otherwise there is no such problem. We propose a general solution which applies also to other modal translations of default logic[5]: e.g. [Nie91] examined below.

Definition 5.3 Let (W, Δ) be a default theory. For each default $\delta = \frac{a:b}{c} \in \Delta$, we introduce a new propositional symbol δ^p, and we define $sepj(\delta) = \frac{a : b \wedge \delta^p}{c}$ and $sepj(\Delta) = \{sepj(\delta) \ / \ \delta \in \Delta\}$.

We call $(W, sepj(\Delta))$ a *separately justified* default theory.

Property 5.3 (W, Δ) and $(W, sepj(\Delta))$ have exactely the same set of R-extensions (respectively, Ł-extensions, K-extensions, C-extensions, pre-extensions, justified pre-extensions).

Indeed, as the new propositional symbols δ^p are all distinct and never appear elsewhere than in $sepj(\delta)$, we cannot have $\neg\delta^p$, thus the defaults δ and $sepj(\delta)$ are strictly equivalent. Thus, considering only the separately justified default theories is not a loss of generality. Here is what happens with the separately justified versions of example 5.1:

Example 5.2 $W = \emptyset$, $sepj(\Delta) = \{\delta_1 = \frac{: b \wedge \delta_1^p}{c}, \delta_2 = \frac{: b \wedge \delta_2^p}{\neg c}\}$.

No change concerning R-extensions, Ł-extensions or K-extensions.

$\mathcal{T}^M = \{H(b \wedge \delta_1^p) \Rightarrow Lc, H(b \wedge \delta_2^p) \Rightarrow L\neg c\}$ which now gives two S-extensions:

– We put $H(b \wedge \delta_1^p) \in E_1'^M$, then we have $Lc \in E_1'^M$, thus $\neg L\neg c \in E_1'^M$ and $\neg H(b \wedge \delta_2^p) \in E_1'^M$, this giving our first S-extension $E_1' = Th(c)$.

– We put $H(b \wedge \delta_2^p) \in E_2'^M$, which gives a second and last S'-extension $E_2' = Th(\neg c)$.

Now, starting from $W' = \{b_1 \Leftrightarrow b_2\}$, $\Delta' = \{\delta_1 = \frac{: b_1 \wedge \delta_1^p}{c}, \delta_2 = \frac{: b_2 \wedge \delta_2^p}{\neg c}\}$,
we get essentially the same result: two S-extensions $E_1' = Th(c)$ and $E_2' = Th(\neg c)$. The controversial behavior of S-extensions seen in example 5.1 cannot occur with separately justified default theories.

[5] A newer version [SS94] gives a slightly different definition in which the operator $\neg H \neg$ respects also the axiom schema of consequence and the axiom schema of Barcan. Thus, whenever $b_1 \Leftrightarrow b_2$ is a tautology, we have also $Hb_1 \Leftrightarrow Hb_2$. But when $W' \Rightarrow (b_1 \Leftrightarrow b_2)$, we still cannot conclude $Hb_1 \Leftrightarrow Hb_2$, which means that the strange behavior in this example is reproduced. Our solution solves the problem with this variant of S-extensions also. The differences of behavior between the two definitions (in [SS91] and [SS94] respectively) seem to be marginal and it is simpler to use [SS91]'s version.

Property 5.4 Every S-extension is a justified pre-extension (converse false).

For a separately justified default theory, every Ł-extension is a S-extension(converse false).

Thus, for separately justified default theories, the notion of "default proof" is the same for S-extensions and for Ł-extensions.

We give only a sketch of the proof:

– Let E be a S-extension. We use property 2.3, defining:

$$\begin{aligned} E_0 &= Th(W) \\ E_{i+1} &= Th(E_i \cup Con(\Delta_i)) \text{ where} \\ & \quad \Delta_i = \{\delta \in \Delta \ / \ Pre(\delta) \in E_i, Con(\delta) \in E \text{ and } \neg Jus(\delta) \notin E\}. \end{aligned}$$

All we have to do is to establish $E = \cup E_i$, which comes from the definition of S-extensions.

– Let E be a Ł-extension with its corresponding F. We define a modal S-extension E^M which is the smallest set closed for Th_T which contains W and verifies also, for any $\delta = \frac{a : b \wedge \delta^p}{c} \in \Delta$:

if $(b \wedge \delta^p) \in F$, $c \in E$ and $a \in E$, then $H(b \wedge \delta^p) \in E^M$,

if $\neg(b \wedge \delta^p) \notin E$ and $a \notin E$, then $H(b \wedge \delta^p) \in E^M$,

if $\neg(b \wedge \delta^p) \notin E$ and $c \in E$, then $H(b \wedge \delta^p) \in E^M$,

otherwise, $\neg H(b \wedge \delta^p) \in E^M$,

We must check that E^M is maximally consistent with respect to the set $\{H(b \wedge \delta^p) \ / \ \delta \in \Delta\}$. This is done thanks to property 3.3.

– Any example in which W is not a S-extension of (W, Δ) establishes that the converse of the first part is false. The following example establishes that the converse of the second part is false.

Example 5.3 $W = \{\neg d\}, \Delta = \{\delta_1 = \frac{: b \wedge \delta_1^p}{c}, \delta_2 = \frac{c : b' \wedge \delta_2^p}{d}\}$.

There is no R-extension here because $\neg d \in W$, $\neg b \notin W$ and $\neg b' \notin W \cup \{c\}$.

$H(b \wedge \delta_1^p) \Rightarrow Lc \in \mathcal{T}^M$, $Lc \wedge H(b' \wedge \delta_2^p) \Rightarrow Ld \in \mathcal{T}^M$, $L\neg d \in \mathcal{T}^M$.

Also, in $\mathcal{T}^M$, we have: $L\neg d \Rightarrow \neg Ld$ (property of system T), thus $H(b' \wedge \delta_2^p) \Rightarrow \neg H(b \wedge \delta_1^p)$. There are two modal S-extensions:
$E^M = Th_T(\mathcal{T}^M \cup \{H(b \wedge \delta_1^p)\})$ and $E'^M = Th_T(\mathcal{T}^M \cup \{H(b \wedge \delta_2^p)\})$. Thus
$\{H(b \wedge \delta_1^p), \neg H(b' \wedge \delta_2^p), L\neg d, Lc\} \subset E^M$ and $\{H(b' \wedge \delta_2^p), \neg H(b \wedge \delta_1^p), L\neg d\} \subset E'^M$, which gives as our two S-extensions: $E = Th(\neg d, c)$ and $E' = Th(\neg d)$.

It is easy to check that there is only one Ł-extension (or K-extension) which is E, corresponding to $F = \{b\}$.

This example shows that there are even more non maximal S-extensions than Ł-extensions.

Now that we have established how closely related are S-extensions and Ł-extensions (they give rise to the same notion of "default proof" and they often coincide), it remains to compare the S-test to our previous tests. In fact we may say that
S-test is a translation of our familiar stability test in [SS91]'s formalism, at least for separately justified default theories.

Here are the main parts of the proof of this affirmation:

Here is a word to word translation of the stability test in [SS91]' terms:

$$\text{for all } \frac{a_k : (b_k \wedge \delta_k^p)}{c_k} \in \Delta,\ (La_k \notin E^M \text{ or } Lc_k \in E^M \text{ or } L\neg(b_k \wedge \delta_k^p) \in E^M). \qquad (1)$$

Now, from the small number of properties of H, and from the definition of modal S-extensions, we may have $\neg H(b_i \wedge \delta_i^p) \in E^M$ only if
$W \cup \{H(b_k \wedge \delta_k^p) : H(b_k \wedge \delta_k^p) \in E^M \text{ and } k \neq i\} \vdash_T \neg H(b_i \wedge \delta_i^p)$. $\neg H(b_i \wedge \delta_i^p)$ may be forced only using $L(\neg b_i \vee \neg \delta_i^p) \Rightarrow \neg H(b_i \wedge \delta_i^p)$ (axiom schema $H_i\phi \Rightarrow \neg L\neg\phi$) or $La_i \wedge \neg Lc_i \Rightarrow \neg H(b_i \wedge \delta_i^p)$ (translation of $\delta_i = \frac{a_i : b_i}{c_i} \in \Delta$). Also, $L(\neg b_i \vee \neg \delta_i^p)$ may be entailed only by $L\neg b_i \vee L\neg\delta_i^p$, that is by $L\neg b_i$, as nothing may entail $L\neg\delta_i^p$. Thus, $\neg H(b_i \wedge \delta_i^p)$ may be entailed if and only if $L\neg b_i \vee (La_i \wedge \neg Lc_i)$ is entailed, and the S-test may be rewritten:

$$\text{for all } \frac{a_i : b_i}{c_i} \in \Delta,\ \text{if } (L\neg b_i \vee La_i \in E^M \text{ and } L\neg b_i \vee \neg Lc_i \in E^M) \text{ then } L\neg b_i \in E^M. \quad (2)$$

– Let us suppose that the stability test succeeds for E^M, that is (1). Also we suppose $(L\neg b_i \vee La_i \in E^M)$ and $(L\neg b_i \vee \neg Lc_i \in E^M)$. Now, from the properties of system T and the definition of E^M we get that $L\neg b_i \vee La_i \in E^M$ if and only if $(L\neg b_i \in E^M$ or $La_i \in E^M)$. Thus we have $(L\neg b_i \in E^M$ or $La_i \in E^M)$ and $(L\neg b_i \vee \neg Lc_i \in E^M)$. If $L\neg b_i \notin E^M$, we must have $La_i \in E^M$ and then (1) $Lc_i \in E^M$, which contradicts $(L\neg b_i \vee \neg Lc_i \in E^M)$. This establishes $L\neg b_i \in E^M$, that is (2): S-test succeeds for E^M.

– Let us suppose now that S-test succeeds for E^M. We suppose also that $La_i \in E^M$ and $Lc_i \notin E^M$ (cf (1)). Then $\neg Hb_i \in E^M$ as otherwise we would get $Lc_i \in E^M$ because $(La_i \wedge H_ib_i \Rightarrow Lc_i) \in E^M$. From S-test we get $L\neg b_i \in E^M$, thus (1), E^M succeeds with the translation of the stability test in [SS91]'s formalism. This establishes the equivalence between the S-test and the stability test for separately justified default theories.

We end our comparison with the existing literature by another translation of defaults into modal logic [Nie91]. This text is interesting because it makes an explicite reference to the "constructivity" of the extensions obtained, moreover the method given is rather natural and elegantly formulated.

Niemelä does not accept individual variables and uses a language $\mathcal{L}_L$ (propositional language $\mathcal{L}$ augmented by a modal operator L). The consequence relation $\models_L$, as in standard *auto-epistemic logic* of [Moo85], is the classical logic entailment, each formula $L\phi$ being treated as a new propositional symbol. For any set of formulas $\Sigma \subseteq \mathcal{L}_L$, we define $Th_L(\Sigma) = \{\phi \in \mathcal{L}_L \ / \ \Sigma \models_L \phi\}$. Likewise the operator H in [SS91], here L has a priori no more properties.

Definitions 5.4 [Nie91, Nie92][6] $\varepsilon = \phi_0, \phi_1, \phi_2, \cdots$ is a "L-hierarchic" enumeration of all the formulas in $\mathcal{L}_L$, *L-hierarchic* meaning that if $L\phi_i$ is a sub-formula of ϕ_j, then $i < j$.

[6]There are some minor differences between these two texts, the definition given here is the more recent one. Also the referenced texts do not explicitely split the definition into its constructive part (pre-extension) and the final test, thus, there are some superficial differences with the original presentations.

$\mathcal{T} = (W, \Delta)$, gives rise to $\mathcal{T}^L = W \cup \{(La \wedge L\neg L\neg b \Rightarrow c) \;/\; \frac{a\,:\,b}{c} \in \Delta\}$ [7] (Δ must be finite).

A set $E_L^\varepsilon(\mathcal{T}^L)$ of modal formulas is a *modal N-extension* of $\mathcal{T}^L$, corresponding to ε, if and only if $E_L^\varepsilon(\mathcal{T}^L) = \bigcup_{i=0}^{\infty} E_i^\varepsilon(\mathcal{T}^L)$ where:

$$
\begin{array}{rcll}
E_0^\varepsilon(\mathcal{T}^L) & = & Th_L(\mathcal{T}^L); & \\
E_{i+1}^\varepsilon(\mathcal{T}^L) & = & Th_L(E_i^\varepsilon(\mathcal{T}^L) \cup \{L\phi_i\}) & \text{if } \phi_i \in E_i^\varepsilon(\mathcal{T}^L), \\
 & & Th_L(E_i^\varepsilon(\mathcal{T}^L) \cup \{\neg L\phi_i\}) & \text{otherwise.}
\end{array}
$$

The name *N-pre-extension* will be used to designate the objective subset $E^\varepsilon(\mathcal{T}^L) = E_L^\varepsilon(\mathcal{T}^L) \cap \boldsymbol{\mathcal{L}}$.

Property 5.5 [Nie91, Nie92] A N-pre-extension $E^\varepsilon(\mathcal{T}^L)$ is a R-extension of $\mathcal{T}$ if and only if $E_L^\varepsilon(\mathcal{T}^L)$ respects the additional condition:

There is no index $i \in \mathbb{N}$ and no sentence $\phi \in \boldsymbol{\mathcal{L}}_L$ such that $\neg L\phi \in E_i^\varepsilon(\mathcal{T}^L)$ and $\phi \in E_i^\varepsilon(\mathcal{T}^L)$. We call this test the N-test. Such a N-pre-extension is called acceptable.

Moreover any R-extension can be obtained in this way.

There are some analogies with [TT92]'s proposal. However, similarly to the S-extensions, and contrary to the σ-constructive extensions, we are guaranteed that **any justified pre-extension is a N-pre-extension** only in the case of separately justified default theories[8].

Any N-pre-extension is a pre-extension.

The converses of these last two results are false. And like with the σ-constructive extensions again, the final test is much more complicated than a stability test, as it is a test for justification and stability together. Indeed, testing for each formula ϕ with $\neg\phi \in Jus(\Delta')$, where Δ' is a generator set of the N-pre-extension E considered, is testing whether E is justified or not. Now, making the test for each formula $\phi \in X \cup Y \cup Y_1$ with $X = \{a \;/\; \frac{a\,:\,b}{c} \in \Delta\}$, $Y = \{\neg b \;/\; \frac{a\,:\,b}{c} \in \Delta\}$ and $Y_1 = \{\neg L\neg b \;/\; \frac{a\,:\,b}{c} \in \Delta\}$ is checking whether the justified pre-extension is stable or not.

Note also that the "constructive aspects" of N-pre-extensions suffer from the fact that the N-pre-extensions may be far too big. They contain a lot of "useless" $\neg L\phi$, even if they are incomplete (i.e. not succeeding with the test).

6 Conclusion

We have shown how various presentations of default theories are very similar, not only in their final notion of "extension", but also in the notions of "pre-extension" they first define. This was not obvious while reading the texts as either they do not reference

[7] $\mathcal{T}^L$ is noted $AE_{DL}(\mathcal{T})$ in [Nie92].

[8] Niemelä's proposal has the same kind of problems as [SS91] in example 5.1. Indeed, in "enumerating all the formulas" it is important to decide whether two formulas only distinguished by their spelling are "two occurrences of the same fomula" (thus must appear only once in the enumeration) or two "different formulas" which happen to be equivalent (thus appearing twice in the enumeration). To avoid this problem, it is safer to restrict our attention to separately justified default theories.

each other or, when they do cite another text, they do not give the precise connections. Various definitions of defaults in Reiter's meaning consist in a first "constructive stage", followed by a stability test. The stability test is a very powerful test on its own, which may be applied to sieve a large class of "pre-extensions", producing all the R-extensions and only these. As class of pre-extensions, we may use all the justified pre-extensions (which is a simply defined class of pre-extensions) or m-extensions of [Łuk88, Łuk90] or some refinement of these, called here K-extensions. Also, some modal simulations of default logic were proven to be of this kind: S-extensions of [SS91] are the best example, the final test of [SS91] being equivalent to the stability test.

Some other proposals can be studied with respect to our general framework even if they are of a slightly greater complexity [TT92, Nie91]: the class of "constructive pre-extensions" they get is much larger than the class of our justified pre-extensions and their final test is more complicated, being a test for justifiability and stability. Lastly, let us note here that other approaches exist which do not fit into our frame, although they use a similar way. For example, it is possible to define a kind of "pre-extension" which is not grounded but which is justified, and then to add a condition which makes it grounded. This is done e. g. in [SR91], [Mac91] or [Kon88]. A priori this gives "highly unconstructive" definitions.

The method used here to define "extensions" thanks to a constructive step followed by a test, allows to attribute some "degree of constructiveness" to each definition. We may say that the less pre-extensions is the better, and to that respect [KBH92] is the best. Clearly, we should also refine this notion in considering how tractable is the constructive step, and how "as tractable as possible" is the final test, but we think that this classification gives an idea, and also that it provides some clues for improving the efficiency of all these methods.

As a by-product of this work we have established that the notion of "default proof" of a formula (i.e. deciding whether a given formula belongs to an extension) is the same for K-extensions of [KBH92], C-extensions of [GCH90], m-extensions of [Łuk88, Łuk90], S-extensions of [SS91], and justified pre-extensions. We have provided "constructive definitions" for all these extensions.

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