Unifying Various Approaches to Default Logic

Preliminary Report

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Abstract. Default logic is the most popular non monotonic formalism. We present a general framework, defining the classical default extensions as "justified pre-extensions" having a particular property, "stability". These pre-extensions have a constructive definition, contrary to classical extensions, and they give rise to a notion of "formula provable by default" which has appeared yet under several formulations in the literature. Also, we show how a less known proposal of this kind gives a very interesting "semi-constructive" definition of default extensions. Finally our framework applies to some modal translations of defaults, making precise the relations between the stability in default logic and in autoepistemic modal logic.

1 Introduction

The aim of default logic is to express concisely and naturally rules with exceptions. A great amount of literature has been published on the subject since its introduction by Reiter in 1980. Here, we show how the classical fix point definitions may be split into a constructive stage and a final test. [11] and [5] give an interesting variant of the original definitions of [10]. [3] gives a "semi- constructive" definition of extensions. Also, following [4], there are "modal simulations" of default logic [12, 13]. [6] has defined a slightly different notion which appears regularly since, in many proposals, explicitely or not [2, 12]. Here, we show how all these definitions relate together. When the final result is Reiter's extensions, we get an "equivalence" in this meaning, but here we look deeper inside the definitions to show what they have in common. This comparison allows to tell which definition is "the most constructive". However its main interest is to unify these proposals which look so different.

The next section introduces our general framework, with the notion of justified pre-extensions, which possess a "constructive definition". This unifies several proposals already made. Section 3 shows how a semi-constructive definition by [3] is an interesting application of our general framework. Also, this is very close to the well-known Łukaszewicz's variant of default logic which gives the same notion of provability by default as our pre-extensions. Section 4 shows that some modal translations of default logic can profitably be studied in our framework.

2 Pre-extensions in default logic

(W, Δ) is a default theory in the meaning of [10]. W is a set of first order formulas in an enumerable language $\mathcal{L}$, and Δ is a set of *defaults* $\delta = \frac{a : b}{c}$ where a, b and c

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are closed formulas in $\mathcal{L}$. For the sake of simplicity we use only one *justification* b.
Notations: $a = Pre(\delta)$, $b = Jus(\delta)$, $c = Con(\delta)$ and for a set Δ we note $Pre(\Delta) = \{Pre(\delta) : \delta \in \Delta\}$, and the same for $Jus(\Delta)$ and $Con(\Delta)$.

Definition 2.1 [11] A set Δ' of defaults is *grounded in* W when for every default $\delta \in \Delta'$, there exists a finite sequence $(\delta_1, ..., \delta_n)$ in Δ' with $\delta_n = \delta$ such that:
$W \vdash Pre(\delta_1)$, $\quad (W \cup Con(\delta_1, \ldots, \delta_i)) \vdash Pre(\delta_{i+1})$ for $1 \leq i \leq n-1$.

A set E of formulas is *grounded in* (W, Δ) when there exists a subset Δ' of Δ which is grounded in W and such that $E = Th(W \cup Con(\Delta'))$.

Definition 2.2 A set Δ' of defaults is *justified in* W when for every default $\delta \in \Delta'$ we have $\neg Jus(\delta) \notin Th(W \cup Con(\Delta'))$.

A set E of formulas is *justified in* (W, Δ) when there exists a subset Δ' of Δ which is justified in W and such that $Pre(\Delta') \subseteq E$ and $E = Th(W \cup Con(\Delta'))$

Definition 2.3 A set E of formulas in $\mathcal{L}$ is a *pre-extension* of a default theory $T = (W, \Delta)$ iff there exists a sequence E_i with $E = \cup E_i$, $E_0 = Th(W)$ and:
$E_{i+1} = Th(E_i \cup Con(\Delta_i))$ with $\Delta_i = \{\delta \in \Delta : Pre(\delta) \in E_i, Con(\delta) \in E\}$.

A *generator set* of a pre-extension E is any subset Δ' of Δ, grounded in W, such that $E = Th(W \cup Con(\Delta'))$. $\Delta' = \cup \Delta_i$ is a generator set of the pre-extension E.

Property 2.1 A set E of formulas is a pre-extension iff there exists a subset Δ' of Δ, grounded in W and such that $E = Th(W \cup Con(\Delta'))$.

Property 2.2 A set E of formulas is a pre-extension of (W, Δ) iff E is closed for Th, $W \subseteq E$, and $\forall \phi \in E$, $\exists \delta_i$ ($\delta_i \in \Delta$ and $Con(\delta_i) \in E$) for $0 \leq i \leq n$ such that
$W \cup Con(\{\delta_0, \ldots, \delta_{i-1}\}) \vdash Pre(\delta_i)$ for $(0 \leq i \leq n)$, and
$W \cup Con(\{\delta_0, \ldots, \delta_n\}) \vdash \phi$. Then $\Delta' = \bigcup_{\phi \in E} \{\delta_0, \ldots, \delta_n\}$ is a generator set of E.

The proofs are obvious. Our pre-extensions are grounded, we are interested by the pre-extensions which are also justified. A pre-extension is justified in (W, Δ) iff it has a generator set Δ' which is justified in W. We call Δ' a *justified generator set.* [5] calls Δ' a *regular* generator set (instead of *justified*) and calls *universe* what we call a grounded set of defaults. Justified pre-extensions are called *E-sets* in [2]. They are called *approachable belief set* in [7], which gives the definition only for normal defaults; [7] gives also the corresponding definition (*m-approachable BJ-pair*) for any kind of default for his own modified default theory [6, 7], described later in this text.

Property 2.3 A set E of formulas in $\mathcal{L}$ is a justified pre-extension of (W, Δ) iff $E = \cup E_i$ with: $E_0 = Th(W)$, $E_{i+1} = Th(E_i \cup Con(\Delta_i))$ where
$\Delta_i = \{\delta \in \Delta : Pre(\delta) \in E_i, Con(\delta) \in E$ and $\neg Jus(\delta) \notin E\}$.
(Proof easy.) This is very close to the original definition of default extensions, here called R-extensions:

Definition 2.4 [10] E is a *R-extension of* (W, Δ) iff $E = \cup E_i$ with: $E_0 = Th(W)$, $E_{i+1} = Th(E_i \cup Con(\Delta_i))$ where $\Delta_i = \{\delta \in \Delta : Pre(\delta) \in E_i$ and $\neg Jus(\delta) \notin E\}$.

A modification of Reiter's definition such as the one appearing in proposition 2.3 has been proposed in [2]. Every R-extension is a justified pre-extension, the converse is false. The problem of determining when a justified pre-extension is a R-extension has been tackled by [5, 11, 12, 3, 1] among others. Thanks to our general framework, we are able to show how closely these texts relate to each other.

Before that, let us notice that we may weaken the notion of *justified*:

Definition 2.5 A *weakly justified* pre-extension E of (W, Δ) is a set of formulas closed for Th, such that $W \subseteq E$ and for every $\phi \in E$, either $W \vdash \phi$, or there exists $\delta_i \in \Delta$ with $Con(\delta_i) \in E$, for $1 \leq i \leq n$, such that $W \cup Con(\delta_1, \ldots, \delta_n) \vdash \phi$, and for $(1 \leq i \leq n)$, $W \cup Con(\delta_1, \ldots, \delta_{i-1}) \vdash Pre(\delta_i)$ and $W \cup Con(\delta_1, \ldots, \delta_n) \not\vdash \neg Jus(\delta_i)$.

Any weakly justified pre-extension is a pre-extension (take $\Delta' = \bigcup_{\phi \in E} \{\delta_1, \ldots, \delta_n\}$ as a generator set). Any justified pre-extension is weakly justified, but the converse is not guaranteed as shown by the following example, inspired by ex. 4.3 in [6]:

Example 2.1 $W = \emptyset$, $\Delta = \{\delta_i = \frac{: q_i}{a \wedge \neg q_1 \wedge \cdots \wedge \neg q_{i-1}}\}_{i=1,2,3,\ldots}$.

$E = Th(a, \neg q_1, \ldots, \neg q_i, \ldots)$ is a weakly justified pre-extension, E is not justified.

It is a pre-extension: in the definition, choose $\Delta_0 = \Delta$, thus $E_1 = Th(W \cup \{a, a \wedge \neg q_1, \ldots\}) = E$. It is a weakly justified pre-extension: $\phi \in E$ iff $(a, \neg q_1, \ldots, \neg q_{i-1}) \vdash \phi$ for some $i \geq 0$ (by compactness), that is iff $W \cup \; Con(\{\delta_i\}) \vdash \phi$, and we also have $W \cup \; Con(\{\delta_i\}) \not\vdash \neg q_i$. E is not a justified pre-extension: We cannot find any finite generator set Δ' (one δ_i would suffice, and $\neg q_i$ would not be in E). For any infinite $\Delta' \subseteq \Delta$, for any $\delta_j \in \Delta'$, there is $\delta_k \in \Delta'$ with $Con(\delta_k) \vdash \neg q_j$.

For any *finite* Δ, there is equivalence between weakly justified and justified pre-extensions. For infinite Δ, it can be argued that the notion of weakly justified pre-extension is the best one.

Property 2.4 A (weakly) justified pre-extension is unconsistent iff W itself is unconsistent. (Proof easy.)

A justified pre-extension always exists. Moreover, it is easy to give a *constructive definition* (not using the final result E in the steps used to define the successive E_i). This affirmation may seem paradoxical as the characteristic property 2.3 seems to be "less constructive" than the definition of R-extensions: the final result E appears twice instead of once only. But here is another equivalent definition:

Property 2.5 E is a justified pre-extension of (W, Δ) iff there exists two sequences $(E_i)_{i \in \mathbb{N}}$, $(F_i)_{i \in \mathbb{N}}$ of sets of formulas such that $E = \cup E_i$, $E_0 = W$, $F_0 = \emptyset$; and:
– If there is some $\delta_i \in \Delta$ such that $(Pre(\delta_i) \in E_i$ and
for any $\phi \in F_i \cup \{Jus(\delta_i)\}$, we have $E_i \cup Con(\delta_i) \not\vdash \neg\phi)$ then
$(E_{i+1} = E_i$ and $F_{i+1} = F_i)$ or $(E_{i+1} = Th(E_i \cup Con(\delta_i))$ and $F_{i+1} = F_i \cup \{Jus(\delta_i)\})$;
if there is no such δ_i then $(E_{i+1} = E_i$ and $F_{i+1} = F_i)$.

(Proof easy.) This reminds some definitions appearing in the literature, e. g. [6, 1], even if it is slightly different. This characterization is not only "constructive" but of a "one default at a time" kind. However, there may be an infinite number of steps and there is undeterminism involved at each step.

We need another property if we want to obtain R-extensions.

Definition 2.6 A set E of formulas satisfying $E = Th(E)$ is *stable for* Δ iff for any default $\delta \in \Delta$: *if* $Pre(\delta) \in E$ *and* $\neg Jus(\delta) \notin E$ *then* $Con(\delta) \in E$.

Checking for stability of a given set E is called *stability test*, or *st-test*.

This property appears in several texts, moreover it is an adaptation of a property appearing in the founding paper [10]. It is clear that any R-extension is a justified pre-extension which is stable. The converse is also true. As this is a relatively well known result, we do not give the proof here. Thus we have split the definition of R-extensions into three properties: groundedness, justification and stability. Groundedness is constructive, and even it is much easier to guarantee this condition in the construction of extensions (however [11] computes R-extensions firstly without taking care of groundedness, and then making a check). Justification can easily be made constructive also: it suffices to take care of all the justifications needed at the previous steps and to respect the new justifications at each step. Stability is the only inherently non constructive property: we need the final result E to ensure stability. If we construct our pre-extensions "from below" (with growing pre-extensions), we have problems with the $\neg Jus(\delta) \notin E$ part, and if we design a construction "from above" (decreasing pre-extensions), we have problems with the $Pre(\delta) \in E$ part.

There have been several "semi-constructive" definitions of R-extensions in the literature, consisting in a first constructive stage, giving some class of pre-extensions, followed by a final "test". Generally, even when it has not been recognized, the final test is either a stability test or a more complicated test including a stability test.

3 Justified pre-extensions and Łukaszewicz defaults

For improving our definition, we may try to give the minimum work to the final stability test, constructing fewer pre-extensions. [3] gives one of the most interesting solutions in this direction. Their constructive stage is simple and natural, and it gives only a restricted sort of justified pre-extensions. We call "K-extensions" their pre-extensions, and show how they relate to the well-known Łukaszewicz' extensions.

Definition 3.1 (adaptation of a definition in [3]) E is a *K-extension* iff there exists two sequences (E_i), (F_i) of sets of formulas with $E = \cup E_i$, $E_0 = W$, $F_0 = \emptyset$, and
- for any ordinal $\alpha \neq 0$, we define $E'_\alpha = \underset{i<\alpha}{\cup} E_i$, $F'_\alpha = \underset{i<\alpha}{\cup} F_i$;
if there is some $\delta_\alpha \in \Delta$ such that $E'_\alpha \vdash Pre(\delta_\alpha)$, $E'_\alpha \nvdash Con(\delta_\alpha)$ and
for any $\phi \in F'_\alpha \cup \{Jus(\delta_\alpha)\}$, $E'_\alpha \cup \{Con(\delta_\alpha)\} \nvdash \neg\phi$
then choose one such δ_α and $E_\alpha = Th(E'_\alpha \cup \{Con(\delta_\alpha)\})$, $F_\alpha = F'_\alpha \cup \{Jus(\delta_\alpha)\}$;
if there is no such δ_α then $E_\alpha = E'_\alpha$ and $F'_\alpha = F_\alpha$.

The only two real differences with justified pre-extensions (prop. 2.5) are that if we can increase E_i here we must do it and there is the additional condition $E_i \nvdash Con(\delta)$ (only defaults of some utility are used). Thus K-extensions are justified pre-extensions, the converse being not guaranteed. An apparent third difference is that here we must consider ordinals greater than ω, in order to be sure that we do not stop too early. This complication is not needed in property 2.5 as anyway with justified pre-extensions we may "stop too early". Due to the "constructive nature" of definition 3.1, we get that any default theory has a K-extension.

Property 3.1 [3] A K-extension is a R-extension of (W, Δ) iff it is stable.

K-extensions are very close to the m-extensions of [6, 7], here called Ł-extensions:
Definition 3.2 [6] S and U being two sets of formulas, $\Gamma_1(S,U)$ and $\Gamma_2(S,U)$ are the smallest sets verifying: 1) $\Gamma_1(S,U) = Th(\Gamma_1(S,U))$; 2) $W \subseteq \Gamma_1(S,U)$;
3) for any $\delta \in \Delta$ such that $(Pre(\delta) \in \Gamma_1(S,U))$ and (for any $\phi \in U \cup \{Jus(\delta)\}$, $S \cup \{Con(\delta)\} \nvdash \neg\phi$), then $(Con(\delta) \in \Gamma_1(S,U)$ and $Jus(\delta) \in \Gamma_2(S,U))$.
E is a *Ł-extension* iff there exists a F such that $E = \Gamma_1(E,F)$ and $F = \Gamma_2(E,F)$.

We may characterize Ł-extensions in a way more interesting for us:
Property 3.2 E is a Ł-extension of $\mathcal{T} = (W, \Delta)$ iff there exists a set F and two sequences (E_i), (F_i) of sets of formulas such that $E = \cup E_i$, $F = \cup F_i$ and:
- $E_0 = W$, $F_0 = \emptyset$, $\Delta_0 = \Delta$;
- for any ordinal α other than 0, we define $E'_\alpha = \bigcup_{i<\alpha} E_i$, $F'_\alpha = \bigcup_{i<\alpha} F_i$, $\Delta'_\alpha = \bigcap_{i<\alpha} \Delta_i$,
if there is $\delta_\alpha \in \Delta'_\alpha$ such that $E'_\alpha \vdash Pre(\delta_\alpha)$ and
for any $\phi \in F'_\alpha \cup \{Jus(\delta_\alpha)\}$, $E'_\alpha \cup \{Con(\delta_\alpha)\} \nvdash \neg\phi$
then choose one of these δ_α and define
$E_\alpha = Th(E'_\alpha \cup \{Con(\delta_\alpha)\})$, $F_\alpha = F'_\alpha \cup \{Jus(\delta_\alpha)\}$ and $\Delta_\alpha = \Delta'_\alpha - \{\delta_\alpha\}$;
if there is no such δ_α then $E_\alpha = E'_\alpha$, $F_\alpha = F'_\alpha$ and $\Delta_\alpha = \Delta'_\alpha$.

This is a corollary of theorem 5.18 in [6]. In [6] there is an additional "Ł-stability test" (stability test for Ł-extensions). This test is a rewriting of the condition appearing in each stage, and, as the property is written here, it is unnecessary. Thus, there exists a "constructive" (and even a "one default at a time") definition of Ł-extensions.

The only difference between Ł-extensions and K-extensions is that in definition 3.1 there is the additional requirement $E_i \nvdash Con(\delta)$. Thus any K-extension is a Ł-extension, the converse is not guaranteed as shown by the following example, even if K-extensions and Ł-extensions are rather close together:

Example 3.1 (cf example 4.2 in [6])
$W = a$, $\Delta = \{\delta_1 = \frac{:c}{a}, \delta_2 = \frac{:b}{\neg c}\}$.
First, we look for K-extensions. $E_0 = \{a\}, F_0 = \emptyset$,
δ_1 ? no because $E_0 \vdash a$ (this is the difference with Ł-extensions).
δ_2 ? yes because $E_0 \nvdash \neg c, E_0 \cup \{\neg c\} \nvdash \neg b$. Thus $E_1 = Th(a, \neg c), F_1 = \{b\}$.
There is only one K-extension $E = E_1 = Th(a, \neg c)$. E is also a R-extension.
However there are two Ł-extensions $E' = Th(a)$, associated with $F' = \{c\}$ and
$E = Th(a, \neg c)$, associated with $F = \{b\}$.

The existence of non maximal extensions is a weakness of Łukaszewicz's proposal. Unfortunately, as the following example shows, some non maximal K-extensions are not eliminated by the additional requirement $E_i \nvdash Con(\delta)$ in definition 3.1:
Example 3.2 $W = \emptyset$, $\Delta = \{\delta_1 = \frac{:j_1}{a}, \delta_2 = \frac{:j_2}{a \wedge \neg j_1}\}$.
There are two K-extensions, which are also Ł-extensions:
$E = Th(a)$, with $F = \{j_1\}$, and $E' = Th(a, \neg j_1)$, with $F' = \{j_2\}$.
Note that here st-test eliminates E, E' being the only R-extension.

From property 2.5 it is clear that any Ł-extension is a justified pre-extension.
To continue our comparison with the existing literature, we study here a proposal made in [2]. We have yet signaled that they too introduce justified pre-extensions,

under the name "E-sets". Their goal was to define a "new kind" of default extensions. They came up with a kind of extensions which is very close to Ł-extensions. As this fact is not mentioned in their text, we precise this point now.

Definition 3.3 (adapted from [2]) Let $\mathcal{T} = (W, \Delta)$ be a default theory. Any set E which is a union of justified pre-extensions of $\mathcal{T}$ and which is closed for Th (i.e. $Th(E) = E$) is a *G-pre-extension* of $\mathcal{T}$. Any G-pre-extension of $\mathcal{T}$ which is maximal for $\subseteq$ is a *G-extension* of $\mathcal{T}$.

As justified pre-extensions are closed for Th, the only unions of justified pre-extensions which are closed for Th are the unions $\cup_{i \in I} E_i$, containing an inclusion chain $E_{j_1} \subseteq E_{j_2} \subseteq \cdots$ where $J = \{j_1, j_2, \ldots\} \subseteq I$, such that for any $i \in I$ there exists a $j \in J$ with $E_i \subseteq E_j$. As only the maximal G-pre-extensions will be considered, these unions are needed only with infinite chains of justified pre-extensions as in example 2.1 where $Th(a) \cup Th(a \wedge \neg j_1) \cup Th(a \wedge \neg j_1 \wedge \neg j_2) \cup \cdots$ is a G-extension but is not a Ł-extension. Anyway, the notion of "default proof" is exactly the same with [6] and [2]. This means that a given formula ϕ belongs to a Ł-extension iff it belongs to a G-extension. To see this, it suffices to compare the definition of a *G-proof* (definition 4 in [2]) and the characterization of *Ł-proofs* as given in theorem 7.7 of [6] or theorem 5.126 in [7], they coincide[2]:

Property 3.3 [6, 7] (resp. [2])
A sentence ϕ belongs to a Ł-extension (resp. a G-extension) of (W, Δ) iff there exists a finite sequence $\Delta_0, \cdots, \Delta_k$ of finite subsets of Δ such that: (1) $W \cup Con(\Delta_0) \vdash \phi$; (2) $W \cup Con(\Delta_i) \vdash Pre(\Delta_{i-1})$, $(1 \leq i \leq k)$;
(3) $\Delta_k = \emptyset$; (4) $W \cup \bigcup_{i=0}^{k} Con(\Delta_i) \not\vdash \neg\psi$ for any $\psi \in \bigcup_{i=0}^{k} Jus(\Delta_i)$.

Note that this result holds also with K-extensions, with justified pre-extensions, and with weakly justified pre-extensions instead of Ł- (or G-)extensions. Regarding the respective extensions, here are the precise relations (proofs easy):

Property 3.4 Any maximal Ł-extension is a G-extension. Any G-extension is either a maximal Ł-extension or a union of an infinite chain of Ł-extensions.
G-extensions coincide with maximal weakly justified pre-extensions.

Now, we examine a seemingly different proposal [12], and show that it has several profound analogies with our presention of stable justified pre-extensions.

4 Pre-extensions in modal translations of defaults

Several authors ([4, 9, 13]) have "translated" default logic into a modal formalism. [12] chooses the modal system T:

Definition 4.1 The modal system T adds to classical first order logic a modal operator L (also M is $\neg L \neg$) with the following properties: – axiom schema of necessity: $L\phi \Rightarrow \phi$; consequence: $L(\phi \Rightarrow \psi) \Rightarrow (L\phi \Rightarrow L\psi)$; Barcan: $\forall x (L\phi(x)) \Rightarrow L(\forall x \phi(x))$; – rule of necessitation: If $\vdash \phi$ then $\vdash L\phi$ (if ϕ is a tautology, so is $L\phi$).

[2] Note however that this was not readily apparent due to a mistake in theorem 7.7 in [6], not corrected before [7].

The system of [12] contains also a set of modal operators $\{H_\delta\}_{\delta\in\Delta}$ which respect only one property: $H_\delta\phi \Rightarrow \neg L\neg\phi$ (for any index δ and formula ϕ). Any term $H_\delta\phi$ is considered as another propositional variable. Here is how the authors translate a default theory $\mathcal{T} = (W, \Delta)$ (where W is supposed consistent):

Definition 4.2 ([12] adapted) $\mathcal{T}$ gives rise to $\mathcal{T}^M = \{L\phi : \phi \in W\} \cup \{La \wedge H_\delta b \Rightarrow Lc : \delta = \frac{a:b}{c} \in \Delta\}$. A consistent set E^M of modal formulas is a *modal S-extension* of $\mathcal{T}^M$ iff $\mathcal{T}^M \cup \{H_\delta b \Rightarrow \neg L\neg b : \delta = \frac{a:b}{c} \in \Delta\} \subseteq E^M$ and
1) For any default $\delta = \frac{a:b}{c} \in \Delta$, either $H_\delta b \in E^M$ or $\neg H_\delta b \in E^M$.
2) $Th_T(E^M) = E^M$: E^M is closed for the consequence relation of system T.
3) It is not possible to add another $H_\delta b$ to E^M without provoking an inconsistency.
4) E^M is a minimal set with these properties.

We are interested by the objective formulas (without modality) ϕ such that $L\phi \in E^M$. The name *S-extension* refers to this objective subset E of E^M.

Property 4.1 [12] A S-extension E is a R-extension iff E^M respects the additional condition: *For any* $\delta = \frac{a:b}{c} \in \Delta$, *if* $\neg H_\delta b \in E^M$, *then* $L\neg b \in E^M$.
We call this test the *S-test*. Moreover any R-extension can be obtained in this way.

In the original definitions there was only one operator H. Here is a strange behavior of the *S'-extensions* so defined (definition 4.2 with $H_i\phi \equiv H_j\phi$ for any i, j):
Example 4.1 $W = \emptyset$, $\Delta = \{\delta_1 = \frac{:b}{c}, \delta_2 = \frac{:b}{\neg c}\}$.

The translation respecting the original definition of [12] gives $\mathcal{T}^M = \{Hb \Rightarrow Lc, Hb \Rightarrow L\neg c\}$. As $\mathcal{T}^M \subseteq E^M$ and $E^M = Th_T(E^M)$, we get $(\neg Lc \vee \neg L\neg c) \Rightarrow \neg Hb \in E^M$. Now, $Lc \Rightarrow c$ and $L\neg c \Rightarrow \neg c$ are axioms of the system T, thus $\neg Lc \vee \neg L\neg c \in E^M$. This gives: $\neg Hb \in E^M$. Thus, the only modal S'-extension E^M of $\mathcal{T}^M$ is $Th_T(\mathcal{T}^M)$, no default is applicable: $E = Th(W) = Th(\emptyset)$.

Imagine now that we start from: $W' = \{b_1 \Leftrightarrow b_2\}$, $\Delta' = \{\delta_1 = \frac{:b_1}{c}, \delta_2 = \frac{:b_2}{\neg c}\}$. $\mathcal{T}'^M = \{Hb_1 \Rightarrow Lc, Hb_2 \Rightarrow L\neg c\}$, giving two S'-extensions:
– We put $Hb_1 \in E_1'^M$, then we have $Lc \in E_1'^M$, thus $\neg L\neg c \in E_1'^M$ and $\neg Hb_2 \in E_1'^M$, this giving our first S'-extension $E_1' = Th(c)$.
– We put $Hb_2 \in E_2'^M$. This gives a second and last S'-extension $E_2' = Th(\neg c)$.

We consider this behavior as very contestable: these two default theories should have the same extensions. With our definition 4.2 we get the same result in each case: Both (W, Δ) and (W', Δ') have two S-extensions, $Th(c)$ and $Th(\neg c)$. Note that none S'-extension succeeeds to the S-test, there is no R-extension.

Our solution is very close to the original text, a difference appears only when two defaults have the same justification. Property 4.1 is true with S-extensions and with S'-extensions. Also, S-extensions as well as S'-extensions are justified pre-extensions. There are some analogies between S-extensions and Ł-extensions. In example 4.1 the two S-extensions are also the two Ł-extensions, and the two K-extensions.

Property 4.2 Every Ł-extension is a S-extension (converse false).
Every S-extension is a justified pre-extension (converse false). Thus, the notion of "default proof" is the same for S-extensions and for Ł-extensions.

The proof is not difficult and appears in [8]. Any example in which W is not a S-extension of (W, Δ) establishes that the converse of the second part is false. The following example establishes that the converse of the first part is false.

Example 4.2 $W = \{\neg d\}, \Delta = \{\delta_1 = \frac{:b}{c}, \delta_2 = \frac{c:b'}{d}\}$. We note H_i for H_{δ_i}.

$H_1 b \Rightarrow Lc \in \mathcal{T}^M$, $Lc \wedge H_2 b' \Rightarrow Ld \in \mathcal{T}^M$, $L\neg d \in \mathcal{T}^M$. Also, $L\neg d \Rightarrow \neg Ld \in \mathcal{T}^M$, thus $H_2 b' \Rightarrow \neg H_1 b \in \mathcal{T}^M$. There are two modal S-extensions $E^M = Th_T(\mathcal{T}^M \cup \{H_1 b\})$ and $E'^M = Th_T(\mathcal{T}^M \cup \{H_2 b'\})$, which gives as our two S-extensions $E = Th(\neg d, c)$ and $E' = Th(\neg d)$. There is only one Ł-extension (or one K-extension) E, corresponding to $F = \{b\}$ (there is no R-extension). This example shows that there are even more non maximal S-extensions than Ł-extensions.

Now that we have established how closely related are S-extensions and Ł-extensions (they give rise to the same notion of "default proof" and they often coincide), it remains to compare the S-test to our previous tests. In fact we may say that *S-test is a translation of our familiar stability test in [12]'s formalism.*

Here are the main lines of the proof, (1) is equivalent to (2):

Here is a word to word translation of the stability test in [12]' terms:

$$\text{For any } \delta = \frac{a:b}{c} \in \Delta, \text{ we have } (La \notin E^M, \text{ or } Lc \in E^M, \text{ or } L\neg b \in E^M) \tag{1}$$

$\neg H_{\delta'} b' \in E^M$ only if $W \cup \{H_\delta b : H_\delta b \in E^M$ and $\delta \neq \delta'\} \vdash_T \neg H_{\delta'} b'$ and $\neg H_{\delta'} b'$ may be entailed iff $L\neg b' \vee (La' \wedge \neg Lc')$ is entailed. The S-test may be rewritten: for any

$$\frac{a':b'}{c'} \in \Delta, \text{ if } (L\neg b' \vee La' \in E^M \text{ and } L\neg b' \vee \neg Lc' \in E^M) \text{ then } L\neg b' \in E^M \tag{2}$$

We end our comparison of various presentations of default logic with the way used in [13] to define extensions thanks to a modal logic. The method has some analogies with the one in [12]. We restrict our attention to system T ([13] considers several systems, including T). [13] uses a propositional language $\mathcal{L}$, and $\mathcal{L}_L$ wich is $\mathcal{L}$ augmented by the modal operator L. [13] uses a system T, with the difference, emphasized in [12], that the notion of consequence in T, noted Cn_T, uses the *strong* necessity rule: if $\phi \in E$, then $L\phi \in Cn_T(E)$.

Definitions 4.3 (see [13]) A set E of formulas in $\mathcal{L}_L$ is a *T-expansion of a set of formulas I* iff: $E = Cn_T(I \cup \{\neg L\phi : \phi \notin E\})$. A set E of formulas in $\mathcal{L}_L$ is a *ground T-expansion of a set of formulas I* iff: $E = Cn_T(I \cup \{\neg L\phi : \phi \in \mathcal{L} - E\})$.

A set of formulas E in $\mathcal{L}_L$ is *m-stable* if it is closed for Cn_T and if for every $\phi \notin E$ we have $\neg L\phi \in E$. T-expansions and ground T-expansions are m-stable.

For any set S of formulas in $\mathcal{L}$, there is exactly one m-stable set E such that $E \cap \mathcal{L} = Th(S)$. This unique m-stable set is noted $St(S)$.

The notion of *m-stability* introduced here is the standard notion of stability in modal *autoepistemic logic* (see [4, 9]). Here is how [13] translates a default theory:

Notation 4.4 [13] For any default $\delta = \frac{a:b}{c} \in \Delta$, $emb(\delta) = La \wedge L\neg L\neg b \Rightarrow Lc$, $emb(W, \Delta) = \{L\phi : \phi \in W\} \cup \{emb(\delta) : \delta \in \Delta\}$.

Theorem 4.3 [13] E being a consistent set of propositional formulas, closed for propositional consequence, E is a R-extension of $\mathcal{T} = (W, \Delta)$ iff:

$St(E)$ is a T-expansion of $emb(W, \Delta)$ iff:

$St(E)$ is a ground T-expansion of $emb(W, \Delta)$.

We split this definition into a constructive stage and a final test, in a natural way. Again, as final test we get a kind of stability test, which makes precise the relationship between the stability condition in default logic and m-stability.

Definition 4.5 A *T-extension of* $\mathcal{T} = (W, \Delta)$ *justified by* F *(*$F \subseteq \boldsymbol{\mathcal{L}}$*)* is a consistent set E of formulas in $\boldsymbol{\mathcal{L}}_L$ defined by: $E = Cn_T(emb(W, \Delta) \cup \{\neg L \neg \phi : \phi \in F\})$

Theorem 4.4 (corollary of theorem 4.3) A T-extension E of $\mathcal{T}$ justified by F is such that $E \cap \boldsymbol{\mathcal{L}}$ is a R-extension of $\mathcal{T}$ iff we have: if $\phi \in \boldsymbol{\mathcal{L}} - E$, then $\neg L\phi \in E$. This is called here the *T-test.* Moreover, any R-extension can be obtained in this way.

In order to compare the T-test to the stability test, we compare it to the S-test: *if* $\neg Hb \in E$ *then* $L\neg b \in E$ *for any* $b \in Y$, with $Y = \{b : \frac{a : b}{c} \in \Delta\}$, which is: *if* $L\neg b \notin E$ *then* $\neg Hb \notin E$ *for any* $b \in Y$. Due to the properties of S-extensions, it is also equivalent to: *for any* $b \in Y$*, if* $\neg b \in \boldsymbol{\mathcal{L}} - E$ *then* $Hb \in E$, which with [13]'s translation of default theory gives: *for any* $b \in Y$*, if* $\neg b \in \boldsymbol{\mathcal{L}} - E$ *then* $L\neg L \neg b \in E$, This in turn, due to the properties of Cn_T, is equivalent to: *for any* $b \in Y$*, if* $\neg b \in \boldsymbol{\mathcal{L}} - E$ *then* $\neg L \neg b \in E$. This emphasizes the fundamental difference between S-test (and thus stability test) and T-test: the T-test must be made for any $\phi \in \boldsymbol{\mathcal{L}}$, while the S-test has to be done only for any $b \in Y$. The T-test is far harder to execute to that respect. *The stability test is equivalent to a very partial T-test.*
It remains to see whether the T-test can be simplified into an ordinary stability test. The answer is that this can be done, with some caution.

Theorem 4.5 Let $\mathcal{T} = (W, \Delta)$ be a default theory. Let Y be the set of all the justifications $\{b : \frac{a : b}{c} \in \Delta\}$. If a T-extension E of $\mathcal{T}$, justified by F, is such that for any $b \in Y$ we have: if $\neg b \notin E$ then $\neg L \neg b \in E$, then $E \cap \boldsymbol{\mathcal{L}}$ is a R-extension. This is the *restricted T-test.* Moreover, any R-extension can be obtained in this way.

The proof uses our treatment of S-extensions and is left to the reader. Note however that a T-extension succeeding with this test is not necessarily a T-expansion: it is not necessarily m-stable as $\phi \notin E$ does not imply $\neg L\phi \in E$. The condition needed for R-extensions is not m-stability: we need not to restrict our attention to the autoepistemic expansions, the relevant property is *"autoepistemy with respect to* $\neg Y$*"*, i.e. for any $b \in Y$, if $\neg b \notin E$ then $\neg L \neg b \in E$.

The T-test being a kind of "extended stability test", we now look closer to the first phase of our definition. The unfortunate conclusion is that this phase cannot be considered as "constructive", there are cases where T-extensions do not exist.

Example 4.3 $W = \{\neg a\}, \Delta = \{\frac{: \neg a}{a}\}$: $emb(W, \Delta) = \{L\neg a, L\neg La \Rightarrow La\}$. As $L\neg a \in E$, we get $\neg a \in E$ and also $\neg La \in E$, also, due to the properties of Cn_T, we get $L\neg La \in E$ which gives $a \in E$, thus getting a contradiction: $E = \boldsymbol{\mathcal{L}}_L$.

5 Conclusion

Various presentations of default theories are very similar, not only in the final notion of "extensions" they get, but also in the notions of "pre-extension" they first define.

This was not obvious while reading the texts. These definitions consist in a first "constructive stage", followed by a stability test. The stability test is very powerful, it may be applied to sieve a large class of "pre-extensions", producing all the R-extensions and only these. As class of pre-extensions, we may use all the justified pre-extensions (which constitute a simply defined class of pre-extensions) or L-extensions of [6, 7] or some refinement of these, called here K-extensions. Also, some modal simulations of default logic were proved to be of this kind: S-extensions of [12], or even T-extensions defined so as to encompass the method of [13]. This has given the precise relationship between the classical stability test of default logic and some related notions in modal logics. The final test of [12] is equivalent to the stability test. The "autoepistemic test" used in [13] is a stability test extended to more formulas than strictly necessary (when the goal is R-extensions). A R-extension need not correspond to an autoepistemic expansion, all we need is a restricted (and easier to test) kind of autoepistemy.

The method used here allows to attribute some "degree of constructiveness" to each definition. The less pre-extensions is the better, and to that respect [3] is the best. We should refine this notion, but this classification gives an idea. For example, [11]'s method is one of the least constructive in that classification. The method extracted from [13]'s text is still worse as here the test itself is far more complicated. This may give some clues in order to improve efficiency in computing extensions.

As a by-product of our study, we have proved that the notion of "formula provable by default" is exactly the same for K-extensions of [3], G-extensions of [2], Ł-extensions of [6], S-extensions of [12], and justified pre-extensions. Also, all these kinds of extensions have a "constructive" definition.

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