# Alpha Exercises

S. Rajopadhye

**With Answers**

## 1 Change of Basis and Normalization

Consider the ALPHA program that we have seen in class. It has two local variables $X$ and $Y$ both over the domain $D = \{i, j \mid 0 \leq i, j\}$ consiting of the entire positive quadrant. The equations (the braces correspond to the `case-esac` construct) are given below:

$$X = \left\{ \begin{array}{lcl} \{i, j \mid j = i = 0\} & : & 0.(i, j \to) \\ \{i, j \mid i = 0; j > 0\} & : & Y.(i, j \to i, j-1) \\ \{i, j \mid i > 0; j = 0\} & : & X.(i, j \to i-1, j+1) \\ \{i, j \mid i, j > 0\} & : & X.(i, j \to i-1, j+1) * Y.(i, j \to i, j-1)) \end{array} \right. \tag{1}$$

$$Y = \left\{ \begin{array}{lcl} \{i, j \mid j = 0; i \geq 0\} & : & X \\ \{i, j \mid j > 0; i \geq 0\} & : & X * Y.(i, j \to i+1, j-1)) \end{array} \right. \tag{2}$$

Draw (an approprate subset of :-) the extended dependency graph of the program. Recall that $t_1(i, j) = i + j$ is a valid first dimension of a 2-dimensional schedule. We desire to *skew* the graph so that $i$ now represents the first time dimension. This will be accomplished by a change of basis $(i, j \to i + j, j)$ applied to both the variables.

### 1.1 Show transformed program

The new domains now become the image[1] of $D$ by the function $(i, j \to i + j, j)$, i.e., $D' = \{i, j \mid 0 \leq j \leq i\}$. Using the standard rules of COB the new equations become (the changes are in **boldface**).

[1]Remember: to compute the *pre-image* of a polyhedron by a transformation $(z \to Az + a)$, we replace $z$ by $Az + a$ in the constraints defining the polyhedron, and simplify. To compute *image* of a polyhedron by a general transformation $T$ we need to (i) determine its vertices, (ii) take their image by $T$ and (iii) determine the constraints imposed by the new vertices. But if $T$ is unimodular, then the image is just its pre-image by $T^{-1}$, and we use the first method.

$$
X = \left\{ \begin{array}{lcl}
\{i,j \mid j=i=0\} & : & 0.(i,j\rightarrow) \\
\{i,j \mid i=0; j>0\} & : & Y.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}).(i,j\rightarrow i,j-1) \\
\{i,j \mid i>0; j=0\} & : & X.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}).(i,j\rightarrow i-1,j+1) \\
\{i,j \mid i,j>0\} & : & X.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}).(i,j\rightarrow i-1,j+1)* \\
 & & \qquad Y.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}).(i,j\rightarrow i,j-1)
\end{array} \right\}.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}-\mathbf{j},\mathbf{j})
$$

$$
Y = \left\{ \begin{array}{lcl}
\{i,j \mid j=0; i\geq 0\} & : & X.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}) \\
\{i,j \mid j>0; i\geq 0\} & : & X.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j})* \\
 & & \qquad Y.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}).(i,j\rightarrow i+1,j-1)
\end{array} \right\}.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}-\mathbf{j},\mathbf{j})
$$

## 1.2 Normalize it (show intermediate steps)

**Apply Rule 6 (5 instances)**

$$
X = \left\{ \begin{array}{lcl}
\{i,j \mid j=i=0\} & : & 0.(i,j\rightarrow) \\
\{i,j \mid i=0; j>0\} & : & Y.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j}-\mathbf{1},\mathbf{j}-\mathbf{1}) \\
\{i,j \mid i>0; j=0\} & : & X.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}+\mathbf{1}) \\
\{i,j \mid i,j>0\} & : & X.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}+\mathbf{1})* \\
 & & \qquad Y.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j}-\mathbf{1},\mathbf{j}+\mathbf{1})
\end{array} \right\}.(i,j\rightarrow i-j,j)
$$

$$
Y = \left\{ \begin{array}{lcl}
\{i,j \mid j=0; i\geq 0\} & : & X.(i,j\rightarrow i+j,j) \\
\{i,j \mid j>0; i\geq 0\} & : & X.(i,j\rightarrow i+j,j)* \\
 & & \qquad Y.(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}+\mathbf{j},\mathbf{j}-\mathbf{1})
\end{array} \right\}.(i,j\rightarrow i-j,j)
$$

**Rule 9 (2 instances) and then Rule 10 (6 instances)**

$$
X = \left\{ \begin{array}{lcl}
\{\mathbf{i},\mathbf{j} \mid \mathbf{j}=\mathbf{i}=\mathbf{0}\} & : & 0.(i,j\rightarrow)(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}-\mathbf{j},\mathbf{j}) \\
\{\mathbf{i},\mathbf{j} \mid \mathbf{i}=\mathbf{j}>\mathbf{0}\} & : & Y.(i,j\rightarrow i+j-1,j-1).(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}-\mathbf{j},\mathbf{j}) \\
\{\mathbf{i},\mathbf{j} \mid \mathbf{i}>\mathbf{j}=\mathbf{0}\} & : & X.(i,j\rightarrow i+j,j+1).(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}-\mathbf{j},\mathbf{j}) \\
\{\mathbf{i},\mathbf{j} \mid \mathbf{i}>\mathbf{j}>\mathbf{0}\} & : & (X.(i,j\rightarrow i+j,j+1)* \\
 & & \qquad Y.(i,j\rightarrow i+j-1,j+1)).(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}-\mathbf{j},\mathbf{j})
\end{array} \right.
$$

$$
Y = \left\{ \begin{array}{lcl}
\{\mathbf{i},\mathbf{j} \mid \mathbf{j}=\mathbf{0}; \mathbf{i}\geq \mathbf{0}\} & : & X.(i,j\rightarrow i+j,j).(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}-\mathbf{j},\mathbf{j}) \\
\{\mathbf{i},\mathbf{j} \mid \mathbf{j}>\mathbf{0}; \mathbf{i}\geq \mathbf{0}\} & : & (X.(i,j\rightarrow i+j,j)* \\
 & & \qquad Y.(i,j\rightarrow i+j,j-1)).(\mathbf{i},\mathbf{j}\rightarrow \mathbf{i}-\mathbf{j},\mathbf{j})
\end{array} \right.
$$

**Rule 1 (2 instances) and then Rule 6 again (8 instances)**

$$X = \left\{ \begin{array}{lcl} \{i,j \mid j = i = 0\} & : & 0.(\mathbf{i},\mathbf{j} \rightarrow) \\ \{i,j \mid i = j > 0\} & : & Y(\mathbf{i},\mathbf{j} \rightarrow \mathbf{i}-\mathbf{1},\mathbf{j}-\mathbf{1}) \\ \{i,j \mid i > j = 0\} & : & X.(\mathbf{i},\mathbf{j} \rightarrow \mathbf{i},\mathbf{j}+\mathbf{1}) \\ \{i,j \mid i > j > 0\} & : & X.(\mathbf{i},\mathbf{j} \rightarrow \mathbf{i},\mathbf{j}+\mathbf{1}) * Y.(\mathbf{i},\mathbf{j} \rightarrow \mathbf{i}-\mathbf{1},\mathbf{j}+\mathbf{1}) \end{array} \right.$$

$$Y = \left\{ \begin{array}{lcl} \{i,j \mid j = 0; i \geq 0\} & : & X.(\mathbf{i},\mathbf{j} \rightarrow \mathbf{i},\mathbf{j}) \\ \{i,j \mid j > 0; i \geq 0\} & : & X.(\mathbf{i},\mathbf{j} \rightarrow \mathbf{i},\mathbf{j}) * Y.(\mathbf{i},\mathbf{j} \rightarrow \mathbf{i},\mathbf{j}-\mathbf{1}) \end{array} \right.$$

Draw the new dependence graph and get the geometric intuition.

# 2 Scheduling

$$\begin{aligned} t_X(i,j) &= \begin{pmatrix} i+j \\ i \end{pmatrix} \\ t_Y(i,j) &= \begin{pmatrix} i+j \\ i+2j+1 \end{pmatrix} \end{aligned}$$

## 2.1 Verify that the above is a valid schedule

**Solution**

Our program has 6 distinct dependences summarized as follows:

$$\begin{array}{llll} X[i,j] & \text{depends on} & Y[i,j-1] & \text{for} \quad j > 0; i \geq 0 \\ X[i,j] & \text{depends on} & X[i-1,j+1] & \text{for} \quad i > 0; j \geq 0 \\[1ex] Y[i,j] & \text{depends on} & X[i,j] & \text{for} \quad i,j \geq 0 \\ Y[i,j] & \text{depends on} & Y[i+1,j-1] & \text{for} \quad i \geq 0; j > 0 \end{array}$$

We need to show that the following set of constraints (expressed as pseudo-Alpha programs) hold

$$\left\{ \begin{array}{lcl} \{i,j \mid 0 \leq i; 0 < j\} & : & t_Y[i,j-1] \prec t_X[i,j] \\ \{i,j \mid 0 < i; 0 \leq j\} & : & t_X[i-1,j+1] \prec t_X[i,j] \\ \{i,j \mid 0 \leq i,j\} & : & t_X[i,j] \prec t_Y[i,j] \\ \{i,j \mid 0 \leq i; 0 < j\} & : & t_Y[i+1,j-1] \prec t_Y[i,j] \end{array} \right.$$

We substitute the definitions of $t_X$ and $t_Y$ and simplify this to show that it always holds.