INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems
Introduction to SDE's

TP 2 : Numerical simulation of stochastic differential equations

December 14 and 17, 2018

Exercise 1: Brownian motion on the circle

Let $B = (B(t)\,,\, 0 \leq t \leq T)$ be a one–dimensional standard Brownian motion defined on the interval $[0, T]$, with $B(0) = 0$. Consider the two–dimensional (bilinear) SDE

$$X(t) = X(0) - \int_0^t F\, X(s)\, ds + \int_0^t R\, X(s)\, dB(s) \ ,$$

with initial condition $X(0) = (0, 1)$, and with the 2×2 matrices

$$F = \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

Here, the final time is set to $T = 10$.

(i) **Check that this SDE has a unique solution.**

(ii) **Write the Itô formula for the real–valued function $f(x) = |x|^2$ defined on $\mathbb{R}^2$. Conclude that the solution satisfies the invariant: $|X(t)|^2 = 1$ almost surely, for any $0 \leq t \leq T$.**

(iii) **Solve numerically the SDE, using a Euler scheme with step–size h and using simulated sample–paths of the Brownian motion. Repeat the same experiment a few times (with different sample–paths of the Brownian motion in each experiment) and plot the corresponding two–dimensional approximations.**

Try $h = 2^{-6}$ down to $h = 2^{-10}$ for instance.

How much is the invariant satisfied by the approximation?

The objective is now to obtain an accurate numerical approximation of the statistics $\mathbb{E}|X(t)|^2$ (the true value of which is known), using empirical Monte Carlo approximation based on Euler schemes. In practice, given a step–size h and a sample size N, the procedure consists of

- for any $i = 1, \cdots, N$, solve numerically the SDE using a Euler scheme with step–size h, and store the approximate solution $\bar{X}_i^h(t)$ for any $0 \leq t \leq T$,

- compute the empirical mean
$$\frac{1}{N} \sum_{i=1}^{N} |\bar{X}_i^h(t)|^2 \ ,$$
as an estimator (depending on h and N) of $\mathbb{E}|X(t)|^2$ for any $0 \leq t \leq T$.

(iv) **Implement this procedure, and plot the empirical mean as a function of time, for different values of h and N.**

Try $h = 2^{-6}$ down to $h = 2^{-10}$ and $N = 100$, $N = 1000$ up to $N = 10000$ for instance.

How much is the invariant satisfied by the approximation?

(v) **Implement the Romberg–Richardson extrapolation procedure, with two step–size h and $2\,h$.**

Try a step–size as coarse as $h = 2^{-6}$ (hence $2\,h = 2^{-5}$) and a sample–size as large as $N = 10000$ for instance.

Check that the appropriate combination of the empirical mean with step size h and the empirical mean with double step–size $2\,h$ provides a more accurate numerical approximation of the statistics $\mathbb{E}|X(t)|^2$ for any $0 \leq t \leq T$.

[Hint: to produce Brownian motion sample–paths with double step–size $2\,h$, use the same Brownian motion sample–paths already simulated with step–size h.]

(vi) **What about the variance of the three different Monte Carlo approximations with step–size h, with double step–size $2\,h$, and with extrapolation? Display histograms for instance.**

Exercise 2: Stationary Gaussian diffusion

Let $B = (B(t)\,,\, 0 \leq t \leq T)$ be a two–dimensional standard Brownian motion defined on the interval $[0, T]$, with $B(0) = 0$. Consider the two–dimensional (linear) SDE

$$X(t) = X(0) + \int_0^t (-c\,I + R)\,X(s)\,ds + \sigma\,B(t)\ , \tag{$\star$}$$

with two real numbers $c > 0$ and σ, and with the 2×2 matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ .$$

Here, the final time is set to $T = 10$. It is further assumed that the initial condition $X(0)$ has zero mean $\mathbb{E}[X(0)] = 0$ and finite variance $\mathbb{E}[X(0)\,X^*(0)] = \Sigma$.

(i) **Check that this SDE has a unique solution. Show that the solution satisfies $\mathbb{E}[X(t))] = 0$ for any $0 \leq t \leq T$.**

(ii) **Write the Itô formula for the matrix–valued function $f(x) = x\,x^*$ defined on $\mathbb{R}^2$, and give the differential equation satisfied by the covariance matrix $\Sigma(t) = \mathbb{E}[X(t)\,X^*(t)]$.**

[Hint: consider the real–valued process $u^*\,X(t)$, where u is an arbitrary two–dimensional vector, and write the Itô formula for the real–valued function $f(r) = r^2$ defined on $\mathbb{R}$.]

(iii) **Under which condition on c and σ^2, and on the variance Σ at initial time $t = 0$, is the solution stationary (in the following weak sense: $\mathbb{E}[X(t)\,X^*(t)] = \Sigma$ for any $0 \leq t \leq T$)?**

What is the expression of $\mathbb{E}|X(t)|^2$ in this case?

[Hint: the condition is $\sigma^2 = 2\,c\,\kappa^2$ and $\Sigma = \kappa^2\,I$.]

From now on, the following numerical values are used: $c = \frac{1}{2}$, $\sigma = 1$ and $\Sigma = I$.

(iv) **Solve numerically the SDE, using a Euler scheme with step–size h and using simulated sample–paths of the Brownian motion. Repeat the same experiment a few times (with different sample–paths of the Brownian motion in each experiment) and plot the corresponding two–dimensional approximations.**

Try $h = 2^{-6}$ down to $h = 2^{-10}$ for instance.

The objective is now to obtain an accurate numerical approximation of the statistics $\mathbb{E}|X(t)|^2$ (the true value of which is known), using empirical Monte Carlo approximation based on Euler schemes. In practice, given a step–size h and a sample size N, the procedure consists of

- for any $i = 1, \cdots, N$, solve numerically the SDE using a Euler scheme with step–size h, and store the approximate solution $\bar{X}_i^h(t)$ for any $0 \leq t \leq T$,
- compute the empirical mean

$$\frac{1}{N} \sum_{i=1}^{N} |\bar{X}_i^h(t)|^2 ,$$

as an estimator (depending on h and N) of $\mathbb{E}|X(t)|^2$ for any $0 \leq t \leq T$.

(v) **Implement this procedure, and plot the empirical mean as a function of time, for different values of h and N.**

Try $h = 2^{-6}$ down to $h = 2^{-10}$ and $N = 100$, $N = 1000$ up to $N = 100000$ (vs. $N = 10000$ in the previous exercice) for instance.

How much is the invariant satisfied by the approximation?

(vi) **Implement the Romberg–Richardson extrapolation procedure, with two step–size h and $2\,h$.**

Try a step–size as coarse as $h = 2^{-6}$ (hence $2\,h = 2^{-5}$) and a sample–size as large as $N = 100000$ for instance.

Check that the appropriate combination of the empirical mean with step size h and the empirical mean with double step–size $2\,h$ provides a more accurate numerical approximation of the statistics $\mathbb{E}|X(t)|^2$ for any $0 \leq t \leq T$.

[Hint: to produce Brownian motion sample–paths with double step–size $2\,h$, use the same Brownian motion sample–paths already simulated with step–size h.]

(vii) **What about the variance of the three different Monte Carlo approximations with step–size h, with double step–size $2\,h$, and with extrapolation? Display histograms for instance.**