INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems
## Introduction to SDE's

TP 1 : Recursive simulation of a Brownian motion

November 21, 2018

The objective is to simulate sample paths of a Brownian motion on refined subdivisions, and ultimately to construct the Brownian motion via approximation.

Let $B = (B(t)\,,\, 0 \leq t \leq 1)$ be a one–dimensional standard Brownian motion defined on the interval $[0,1]$, with $B(0) = 0$.

A set of $(n+1)$ time–space pairs $((t_i, B_i)\ i = 0, \cdots, n)$ (where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ defines a subdivision of the time interval $[0,1]$ in n sub–intervals) is said *representative* of a Brownian motion sample path iff the random vector $(B_0, B_1, \cdots, B_{n-1}, B_n)$ has the same joint distribution as the random vector $(B(t_0), B(t_1), \cdots, B(t_{n-1}), B(t_n))$.
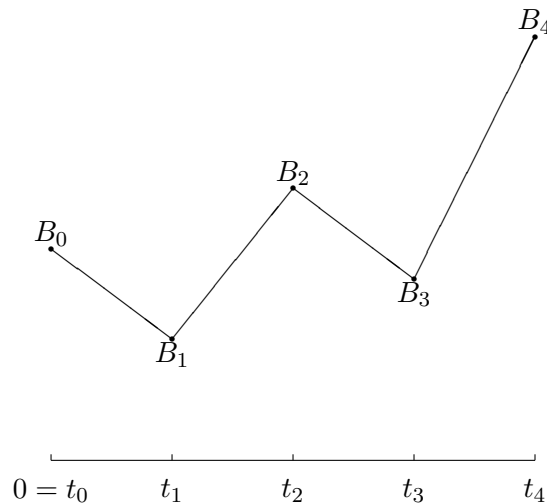

(i) **Given the subdivision $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of the time interval $[0,1]$ in n sub–intervals, explain how the random variables $(B_0, B_1, \cdots, B_{n-1}, B_n)$ could be simulated.**

   Implement this procedure in MATLAB for a uniform subdivision with $n = 8$ sub–intervals, and plot the resulting polygonal approximation.

[Attention: do not (actually, never) use loops indexed by sub–intervals, i.e. loops for i running from 1 to n.]

Assume that a *representative* set of $(n+1)$ time–space pairs $((t_i, B_i)\ i = 0, \cdots, n)$ is available. The next step is to enrich the approximation, i.e. to refine the subdivision by inserting a new time–space pair, such that the resulting set of $(n+2)$ time–space pairs is still *representative* of a Brownian motion sample path. Without loss of generality, assume that a new subdivision time s^* is proposed that belongs to the i–th sub–interval, i.e. $t_{i-1} < s^* < t_i$. How should the random variable B^* be simulated such that the random vector $(B_0, \cdots, B_{i-1}, B^*, B_i, \cdots, B_n)$ has the same joint distribution as the random vector $(B(t_0), \cdots, B(t_{i-1}), B(s^*), B(t_i), \cdots, B(t_n))$?
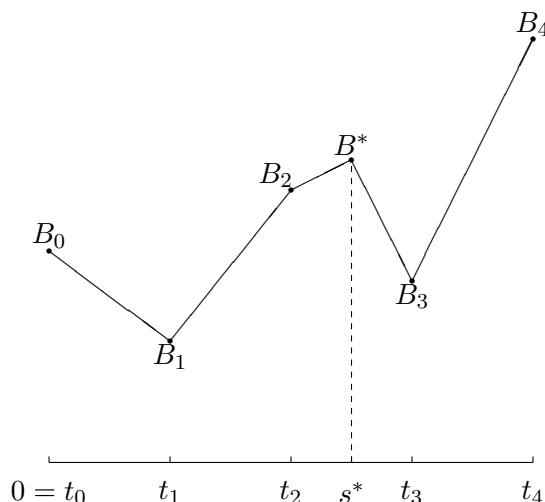
(ii) **List the conditions that the increments $(B^* - B_{i-1})$ and $(B_i - B^*)$ should satisfy.**

(iii) **Characterize the probability distribution of the random variable B^* that would satisfy these conditions.**

[Hint: Without loss of generality, set

$$B^* = B_{i-1}\, \frac{t_i - s^*}{t_i - t_{i-1}} + B_i\, \frac{s^* - t_{i-1}}{t_i - t_{i-1}} + Z \ ,$$

and solve for the probability distribution of the random variable Z.]

(iv) **With the solution of this problem, formulated and solved locally on the sub–interval $[t_{i-1}, t_i]$ only, check that the other conditions that the random variable B^* should satisfy globally, w.r.t. the whole *representative* set, can be easily satisfied.**

This was the solution for inserting one single new time–space pair. Ideally, starting at level n from a *representative* set of $(n + 1)$ time–space pairs $((t_i, B_i)\ i = 0, \cdots, n)$, one would like to subdivide each of the n sub–intervals, i.e. to insert n new time–space pairs, such that the resulting set of $(2n + 1)$ time–space pairs obtained at level $(n + 1)$ by dichotomy is still *representative* of a Brownian motion sample path. This dichotomy can be implemented sequentially, with one sub–interval subdivided (and the corresponding new time–space pair simulated) after the other.

(v) **Show that (and explain why) this dichotomy can be implemented simultaneously, with sub–intervals subdivided (and the corresponding new time–space pairs simulated) all at the same time.**

Implement this refinement procedure in MATLAB to obtain a uniform subdivision with $n = 16$ sub–intervals.

From now on, it is convenient to denote the time–space pairs associated with the uniform dyadic subdivision of the interval $[0, 1]$ in $n(k) = 2^k$ sub–intervals, as $((t_i^{(k)}, B_i^{(k)}), i = 0, \cdots, n(k))$.

(vi) **Repeat the refinement procedure at any level k to obtain a *representative* set associated with the uniform dyadic subdivision at level k, for $k = 3, \cdots, 15$.**

[Hint: clearly $t_i^{(k)} = i \, 2^{-k}$ for any $i = 0, \cdots, 2^k$ and it follows from the result obtained in response to question (iii) that

$$B_{2i}^{(k)} = B_i^{(k-1)} \qquad \text{and} \qquad B_{2i+1}^{(k)} = \tfrac{1}{2}\,(B_{i+1}^{(k-1)} + B_i^{(k-1)}) + Z_i^{(k)}$$

for any $i = 0, \cdots, 2^{k-1}$, where $(Z_i^{(k)}, i = 0, \cdots, 2^{k-1})$ are i.i.d. Gaussian random variables with zero mean and variance $\tfrac{1}{4}\,(t_{i+1}^{(k-1)} - t_i^{(k-1)}) = 2^{-(k+1)}$.]

[Attention: the index 0 does not exist in MATLAB, hence the notations used in the text should be interpreted differently in the code (in particular, odd (even) indices in the text will become even (odd) indices in the code!).]

[Remark: a loop indexed by levels, i.e. a loop for k running from 3 to 15 is acceptable here.]

(vii) **At each level k, compute the approximate quadratic variation, the approximate Itô integral and the approximate Stratonovich integral, defined as**

$$V_k = \sum_{i=1}^{n(k)} (B(t_i^{(k)}) - B(t_{i-1}^{(k)}))^2 \;,$$

$$I_k = \sum_{i=1}^{n(k)} B(t_{i-1}^{(k)})\,(B(t_i^{(k)}) - B(t_{i-1}^{(k)})) \;,$$

and

$$S_k = \sum_{i=1}^{n(k)} \tfrac{1}{2}\,(B(t_i^{(k)}) + B(t_{i-1}^{(k)}))\,(B(t_i^{(k)}) - B(t_{i-1}^{(k)})) \;,$$

respectively, and plot these three quantities as a function of the level. Compare with the theoretical values that should be obtained in the limit as the level k tends to infinity.