INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems
Introduction to SDE's

TD 3 : Stochastic differential equations and diffusion processes

December 5, 2018

Exercise 1 [Brownian motion on the circle] Let $B = (B(t)\,,\, 0 \leq t \leq T)$ be a one–dimensional standard Brownian motion defined on the interval $[0,T]$, with $B(0) = 0$. Consider the two–dimensional (bilinear) SDE

$$X(t) = X(0) - \int_0^t F\, X(s)\, ds + \int_0^t R\, X(s)\, dB(s) \;,$$

with initial condition $X(0) = (0,1)$, and with the 2×2 matrices

$$F = \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;.$$

(i) **Check that this SDE has a unique solution.**

(ii) **Write the Itô formula for the real–valued function $f(x) = |x|^2$ defined on $\mathbb{R}^2$. Conclude that the solution satisfies the invariant: $|X(t)|^2 = 1$ almost surely, for any $0 \leq t \leq T$.**

Solution

[Solution postponed].

□

Exercise 2 [Stationary Gaussian diffusion] Let $B = (B(t)\,,\, 0 \leq t \leq T)$ be a two–dimensional standard Brownian motion defined on the interval $[0,T]$, with $B(0) = 0$. Consider the two–dimensional (linear) SDE

$$X(t) = X(0) + \int_0^t (-c\, I + R)\, X(s)\, ds + \sigma\, B(t) \;,$$

with two real numbers $c > 0$ and σ, and with the 2×2 matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;.$$

It is further assumed that the initial condition $X(0)$ has zero mean $\mathbb{E}[X(0)] = 0$ and finite variance $\mathbb{E}[X(0)\, X^*(0)] = \Sigma$.

(i) **Check that this SDE has a unique solution. Show that the solution satisfies $\mathbb{E}[X(t))] = 0$ for any $0 \leq t \leq T$.**

(ii) **Write the Itô formula for the matrix–valued function $f(x) = x\, x^*$ defined on $\mathbb{R}^2$, and give the differential equation satisfied by the covariance matrix $\Sigma(t) = \mathbb{E}[X(t)\, X^*(t)]$.**

[Hint: consider the real–valued process $u^*\, X(t)$, where u is an arbitrary two–dimensional vector, and write the Itô formula for the real–valued function $f(r) = r^2$ defined on $\mathbb{R}$.]

Solution

[Solution postponed].

□

(iii) **Under which condition on c and σ^2, and on the variance Σ at initial time $t = 0$, is the solution stationary (in the following weak sense: $\mathbb{E}[X(t)\, X^*(t)] = \Sigma$ for any $0 \leq t \leq T$).**

Solution

[Solution postponed].

□

Exercise 3 [Wright–Fisher diffusion approximation] Consider the following simplified model for the reproduction of individuals through the transmissions of alleles (alternative types of the same gene). Consider here the case of one gene, with two alleles A and a. The population size N is assumed finite and constant at each generation. At generation k, each individual inherits the allele of its parent, a randomly (uniformly) selected (with replacement) individual present in the population at generation $(k-1)$. Define the random variable X_k^N to be the number of allele of type A present at generation k.

(i) **Show that the random variable X_k^N takes values in $\{0, 1, \cdots, N\}$, and that the sequence $(X_k^N\, ,\, k \geq 0)$ forms a Markov chain with transition probability matrix**

$$\pi_{i,j}^N = \mathbb{P}[X_k^N = j \mid X_{k-1}^N = i] = \begin{pmatrix} N \\ j \end{pmatrix} \, (\frac{i}{N})^j \, (1 - \frac{i}{N})^{N-j} \ ,$$

for any $i, j \in \{0, 1, \cdots, N\}$.

Solution

Conditionally on $X_{k-1}^N = i$, the probability for an individual at generation k of getting an allele A is equal to the proportion $p_i = i/N$ of allele A available at generation $(k-1)$. Therefore, conditionally on $X_{k-1}^N = i$, the random variable X_k^N is a sum of N independent Bernoulli random variables with parameter $p_i = i/N$, i.e. the random variable X_k^N follows a binomial $\mathrm{Bin}(N, i/N)$ distribution. In other words

$$\pi_{i,j}^N = \mathbb{P}[X_k^N = j \mid X_{k-1}^N = i] = \binom{N}{j} \, (\frac{i}{N})^j \, (1 - \frac{i}{N})^{N-j} \ ,$$

for any $i, j \in \{0, 1, \cdots, N\}$.

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(ii) **Check that**

$$\mathbb{E}[X_k^N \mid X_{k-1}^N = i] = i \qquad \textbf{and} \qquad \mathbb{E}[(X_k^N - X_{k-1}^N)^2 \mid X_{k-1}^N = i] = i \, (1 - \frac{i}{N}) \ .$$

---------- Solution ----------

Recall that a binomial $\mathrm{Bin}(N, p)$ random variable has mean $N \, p$ and variance $N \, p \, (1 - p)$, so that

$$\mathbb{E}[X_k^N \mid X_{k-1}^N = i] = N \, \frac{i}{N} = i \qquad \text{hence} \qquad \mathbb{E}[X_k^N - X_{k-1}^N \mid X_{k-1}^N = i] = 0 \ ,$$

and

$$\mathbb{E}[(X_k^N - X_{k-1}^N)^2 \mid X_{k-1}^N = i] = N \, \frac{i}{N} \, (1 - \frac{i}{N}) = i \, (1 - \frac{i}{N}) \ . \qquad \square$$

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Thinking more in term of frequencies (i.e. proportions) rather than in terms of number of individuals, introduce the normalized random variable $Y_k^N = X_k^N / N$.

(iii) **Show that the random variable Y_k^N takes values in $\{0, 1/N, \cdots, 1 - 1/N, 1\} \subset [0, 1]$, and check that**

$$\mathbb{E}[Y_k^N \mid Y_{k-1}^N = p] = p \ ,$$

and

$$\mathbb{E}[(Y_k^N - Y_{k-1}^N)^2 \mid Y_{k-1}^N = p] = \frac{1}{N} \, p \, (1 - p) \ ,$$

for any $p \in \{0, 1/N, \cdots, 1 - 1/N, 1\}$.

---------- Solution ----------

For any $p \in \{0, 1/N, \cdots, 1 - 1/N, 1\}$, there exists some $i \in \{0, 1, \cdots, N\}$ such that $p = i/N$, hence

$$\mathbb{E}[Y_k^N \mid Y_{k-1}^N = p] = \frac{1}{N} \, \mathbb{E}[X_k^N \mid X_{k-1}^N = i] = \frac{i}{N} = p \ ,$$

and

$$\begin{aligned} \mathbb{E}[(Y_k^N - Y_{k-1}^N)^2 \mid Y_{k-1}^N = p] &= \frac{1}{N^2} \, \mathbb{E}[(X_k^N - X_{k-1}^N)^2 \mid X_{k-1}^N = i] \\ &= \frac{1}{N^2} \, i \, (1 - \frac{i}{N}) = \frac{1}{N} \, p \, (1 - p) \ . \end{aligned}$$

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(iv) **Show that the candidate limit (in distribution, as the population size $N \uparrow \infty$) of the continuous–time process interpolating points Y_k^N at time instants $t_k^N = k/N$, is the solution of the SDE**

$$X(t) = X(0) + \int_0^t \sqrt{X(s)\,(1 - X(s))}\, dB(s) \ .$$

Check that there exists a unique solution to this SDE, taking values in $[0, 1]$.

[Hint: extend the definition of the diffusion coefficient outside the interval $[0, 1]$.]

Exercise 4 [Exit time of a one–dimensional diffusion process] Let $B(t)$ be a one–dimensional standard Brownian motion, and consider the SDE

$$X(t) = X(0) + \int_0^t b(X(s))\, ds + \int_0^t \sigma(X(s))\, dB(s) \ ,$$

where the drift and the diffusion coefficients satisfy the global Lipschitz condition and the linear growth condition. Let L denote the associated second–order differential operator. Let $a < c$ and consider the two hitting times

$$T_a = \inf\{t \geq 0\,,\, X(t) = a\} \qquad \text{and} \qquad T_c = \inf\{t \geq 0\,,\, X(t) = c\} \ ,$$

of a and c respectively, and let

$$T_{a,c} = T_a \wedge T_c = \inf\{t \geq 0 \,:\, X(t) \notin (a, c)\} \ ,$$

denote the exit time from the open interval (a, c). Assume that there exist two bounded functions f and g, twice differentiable with bounded first and second derivatives, such that

$$L\, f(x) = 0 \qquad \text{for any } a \leq x \leq c,$$

up to two (multiplicative and additive) arbitrary normalizing constants, and such that

$$L\, g(x) = -1 \qquad \text{for any } a \leq x \leq c,$$

with conditions $g(a) = g(c) = 0$, respectively.

(i) **Show that**

$$\mathbb{E}_{0,x}[T_a \wedge T_c] = g(x) < \infty \qquad \textbf{and} \qquad \mathbb{P}_{0,x}[T_a < T_c] = \frac{f(c) - f(x)}{f(c) - f(a)} \ ,$$

for any starting point $x \in (a, c)$.

Solution

The Itô formula for the function g yields

$$g(X(t)) \;=\; g(X(0)) + \int_0^t [g'(X(s))\, b(X(s)) + \tfrac{1}{2}\, g''(X(s))\, a(X(s))\, ds]$$

$$+ \int_0^t g'(X(s))\, \sigma(X(s))\, dB(s) \ ,$$

for any $t \geq 0$, and in particular for $t \wedge T_{a,c}$

$$g(X(t \wedge T_{a,c})) \;=\; g(X(0)) + \int_0^{t\wedge T_{a,c}} [g'(X(s))\, b(X(s)) + \tfrac{1}{2}\, g''(X(s))\, a(X(s))\, ds]$$

$$+ \int_0^{t\wedge T_{a,c}} g'(X(s))\, \sigma(X(s))\, dB(s)$$

$$= \; g(X(0)) - (t \wedge T_{a,c}) + \int_0^{t\wedge T_{a,c}} g'(X(s))\, \sigma(X(s))\, dB(s) \ .$$

Indeed, note that $a \leq X(s) \leq c$ for any $0 \leq s \leq t \wedge T_{a,c}$ and

$$L\, g(y) = g'(y)\, b(y) + \tfrac{1}{2}\, g''(y)\, a(y) = -1 \ ,$$

for any $a \leq y \leq c$, hence the identity holds in particular for $y = X(s)$ with $0 \leq s \leq t \wedge T_{a,c}$. The integrand belongs to $M^2([0,T])$ so that the stochastic integral has zero expectation, and using the optional sampling theorem for the bounded stopping time $t \wedge T_{a,c}$ yields

$$\mathbb{E}_{0,x}[g(X(t \wedge T_{a,c}))] = g(x) - \mathbb{E}_{0,x}[t \wedge T_{a,c}] \ ,$$

for any starting point $a \leq x \leq c$. Note that

$$0 \leq \mathbb{E}_{0,x}[t \wedge T_{a,c}] = g(x) - \mathbb{E}_{0,x}[g(X(t \wedge T_{a,c}))] \leq M \qquad \text{with} \qquad M = 2 \max_{a \leq y \leq c} |g(y)| \ ,$$

and

$$0 \leq \mathbb{E}_{0,x}[t \wedge T_{a,c}] = t\, \mathbb{P}_{0,x}[T_{a,c} = \infty] + \mathbb{E}_{0,x}[(t \wedge T_{a,c})\, 1_{(T_{a,c} < \infty)}] \leq M \ ,$$

hence the stopping time $T_{a,c}$ is almost surely finite, and using the monotone convergence theorem yields

$$\mathbb{E}_{0,x}[t \wedge T_{a,c}] \to \mathbb{E}_{0,x}[T_{a,c}] \leq M \ ,$$

as $t \uparrow \infty$. By continuity $g(X(t \wedge T_{a,c})) \to g(X(T_{a,c})) = 0$ almost surely as $t \uparrow \infty$. Note that

$$|g(X(t \wedge T_{a,c}))| \leq \max_{a \leq x \leq c} |g(x)| < \infty,$$

is bounded. Using the Lebesgue dominated convergence theorem yields

$$\mathbb{E}_{0,x}[g(X(t \wedge T_{a,c}))] \to 0 \ ,$$

as $t \uparrow \infty$, and uniqueness of the limit yields

$$g(x) = \mathbb{E}_{0,x}[T_{a,c}] \ .$$

The Itô formula for the function f yields

$$f(X(t)) \;=\; f(X(0)) + \int_0^t [f'(X(s))\, b(X(s)) + \tfrac{1}{2}\, f''(X(s))\, a(X(s))\, ds]$$

$$+ \int_0^t f'(X(s))\, \sigma(X(s))\, dB(s) \;,$$

for any $t \geq 0$, and in particular for $t \wedge T_{a,c}$

$$f(X(t \wedge T_{a,c})) \;=\; f(X(0)) + \int_0^{t\wedge T_{a,c}} [f'(X(s))\, b(X(s)) + \tfrac{1}{2}\, f''(X(s))\, a(X(s))\, ds]$$

$$+ \int_0^{t\wedge T_{a,c}} f'(X(s))\, \sigma(X(s))\, dB(s)$$

$$= \; f(X(0)) + \int_0^{t\wedge T_{a,c}} f'(X(s))\, \sigma(X(s))\, dB(s) \;.$$

Indeed, note that $a \leq X(s) \leq c$ for any $0 \leq s \leq t \wedge T_{a,c} \leq T_{a,c}$ and

$$L\, f(y) = f'(y)\, b(y) + \tfrac{1}{2}\, f''(y)\, a(y) = 0 \;,$$

for any $a \leq y \leq c$, and the identity holds in particular for $y = X(s)$ with $0 \leq s \leq t \wedge T_{a,c}$. The integrand belongs to $M^2([0,T])$ so that the stochastic integral has zero expectation, and using the optional sampling theorem for the bounded stopping time $t \wedge T_{a,c}$ yields

$$\mathbb{E}_{0,x}[f(X(t \wedge T_{a,c}))] = f(x) \;,$$

for any starting point $a \leq x \leq c$. Recall that the stopping time $T_{a,c}$ is almost surely finite, hence by continuity $f(X(t \wedge T_{a,c})) \to f(X(T_{a,c}))$ almost surely as $t \uparrow \infty$. Note that

$$|f(X(t \wedge T_{a,c}))| \leq \max_{a \leq x \leq c} |f(x)| < \infty,$$

is bounded. Using the Lebesgue dominated convergence theorem yields

$$\mathbb{E}_{0,x}[f(X(t \wedge T_{a,c}))] \to \mathbb{E}_{0,x}[f(X(T_{a,c}))] \;,$$

as $t \uparrow \infty$, and uniqueness of the limit yields

$$f(x) = \mathbb{E}_{0,x}[f(X(T_a \wedge T_c))] = f(a)\; \mathbb{P}_{0,x}[T_a < T_c] + f(c)\; \mathbb{P}_{0,x}[T_c > T_a] \;.$$

In other words

$$f(x) = f(a)\; \mathbb{P}_{0,x}[T_a < T_c] + f(c)\; (1 - \mathbb{P}_{0,x}[T_a < T_c]) \;,$$

hence

$$\mathbb{P}_x[T_a < T_c] = \frac{f(c) - f(x)}{f(c) - f(a)} \;.$$

Clearly, the solution of the second–order differential equation is defined up to two (multiplicative and additive) arbitrary constants: indeed, if $f(x)$ is a solution, so is $c_1\, f(x) + c_0$. However, no matter which particular solution is considered, the expression is the same, i.e. does not depend on the two arbitrary constants c_0 and c_1.

□

(ii) **Apply these general results to the special case of the (limiting SDE in the) Wright–Fisher genetic model.**

Solution

In this special case, the two boundary points are $a = 0$ and $c = 1$, and the SDE to be considered is

$$X(t) = X(0) + \int_0^t \sqrt{X(s)\,(1 - X(s))}\, dB(s)\ ,$$

i.e.

$$b(x) = 0 \qquad \text{and} \qquad a(x) = x\,(1 - x)\ ,$$

for any $0 \le x \le 1$, and the second–order differential operator is

$$L = \tfrac{1}{2}\, x\,(1 - x)\, \frac{d^2}{dx^2}\ .$$

The first ODE to be considered is

$$L\, f(x) = \tfrac{1}{2}\, x\,(1 - x)\, f''(x) = 0\ ,$$

i.e. $f''(x) = 0$ for any $x \notin \{0, 1\}$. Any first–order polynomial, say $f(x) = c_1\, x + c_0$, is a solution and it follows that

$$\mathbb{P}_{0,x}[T_0 < T_1] = \frac{f(1) - f(x)}{f(1) - f(0)} = 1 - x\ ,$$

for any $0 \le x \le 1$. Note that this expression does not depend on the two arbitrary constants c_0 and c_1, as expected.

The second ODE to be considered is

$$L\, g(x) = \tfrac{1}{2}\, x\,(1 - x)\, g''(x) = -1 \qquad \text{with } g(0) = g(1) = 0.$$

In other words, introducing $h(x) = \frac{1}{2}\, g'(x)$ it holds

$$h'(x) = -\frac{1}{x\,(1 - x)} = -\frac{1}{x} - \frac{1}{1 - x}\ ,$$

hence

$$\tfrac{1}{2}\, g'(x) = h(x) = -\log x + \log(1 - x) + c_1\ .$$

Setting $\phi_0(x) = x - x\,\log x$ and $\phi_1(x) = \phi_0(1 - x)$, it holds

$$\phi_0'(x) = -\log x \qquad \text{and} \qquad \phi_1'(x) = -\phi_0'(1 - x) = \log(1 - x)\ ,$$

hence

$$\begin{aligned} \tfrac{1}{2}\, g(x) &= \phi_0(x) + \phi_1(x) + c_1\, x + c_0 \\ &= x - x\,\log x + (1 - x) - (1 - x)\,\log(1 - x) + c_1\, x + c_0 \\ &= -x\,\log x - (1 - x)\,\log(1 - x) + c_1\, x + c_0 + 1\ . \end{aligned}$$

The two boundary conditions $g(0) = 0$ and $g(1) = 0$ yield

$$c_0 + 1 = 0 \qquad \text{and} \qquad c_1 + c_0 + 1 = 0 \ ,$$

i.e. $c_1 = c_0 = 0$, hence

$$\mathbb{E}_{0,x}[T_0 \wedge T_1] = g(x) = -2\,[x\,\log x + (1-x)\,\log(1-x)] \ ,$$

for any $0 \le x \le 1$.

□

Exercise 5 [Ornstein–Uhlenbeck process] Let $B(t)$ be a one–dimensional standard Brownian motion, and for any positive real $\beta > 0$ and any real γ, consider the one–dimensional SDE

$$X(t) = X(0) - \beta \int_0^t X(s)\,ds + \gamma\,B(t) \ ,$$

where the initial condition $X(0)$ is square–integrable and independent of the Brownian motion.

(i) **Check that there exists a unique solution to this SDE.**

(ii) **Show that the solution is given explicitly as**

$$X(t) = \exp\{-\beta\,t\}\ X(0) + \gamma \int_0^t \exp\{-\beta\,(t-s)\}\ dB(s) \ .$$

[Hint: use the variation of the constant method.]

Solution

Introducing the process Y defined by $Y(t) = X(t) - \gamma\,B(t)$ for any $t \ge 0$, and in particular $Y(0) = X(0)$ for $t = 0$, yields

$$Y(t) = X(0) - \beta \int_0^t X(s)\,ds = Y(0) - \beta \int_0^t Y(s)\,ds - \gamma\,\beta \int_0^t B(s)\,ds \ ,$$

since $Y(0) = X(0)$. In other words, the process Y satisfies the ODE

$$\dot{Y}(t) = -\beta\,Y(t) - \gamma\,\beta\,B(t) \ .$$

The variation of the constant formula provides an explicit expression for the solution

$$Y(t) = \exp\{-\beta\,t\}\ X(0) - \gamma\,\beta \int_0^t \exp\{-\beta\,(t-s)\}\ B(s)\,ds \ ,$$

hence

$$X(t) = \exp\{-\beta\,t\}\ X(0) + \gamma\,B(t) - \gamma\,\beta \int_0^t \exp\{-\beta\,(t-s)\}\ B(s)\,ds \ ,$$

Note that

$$B(t)\ \exp\{\beta\, t\} = \beta \int_0^t \exp\{\beta\, s\}\ B(s)\, ds + \int_0^t \exp\{\beta\, s\}\, dB(s)\ ,$$

multiplying both sides by $\exp\{-\beta\, t\}$ yields

$$B(t) - \beta \int_0^t \exp\{-\beta\,(t-s)\}\ B(s)\, ds = \int_0^t \exp\{-\beta\,(t-s)\}\, dB(s)\ ,$$

and reporting this expression above yields

$$X(t) = \exp\{-\beta\, t\}\ X(0) + \gamma \int_0^t \exp\{-\beta\,(t-s)\}\, dB(s)\ .$$

□

(iii) **Give the expression of the mean, the variance and the correlation coefficient, defined as**

$$m(t) = \mathbb{E}[X(t)] \qquad \textbf{and} \qquad \sigma^2(t) = \mathbb{E}|X(t) - m(t)|^2$$

and

$$\rho(t,h) = \mathbb{E}[(X(t+h) - m(t+h))\,(X(t) - m(t))]\ ,$$

respectively. Show that for a special choice of γ in terms of $\beta > 0$ and $\sigma^2(0)$, the variance and the correlation coefficient do not depend on $t \geq 0$.

Solution

Recall that

$$X(t) = \exp\{-\beta\, t\}\ X(0) + \gamma \int_0^t \exp\{-\beta\,(t-s)\}\ dB(s)\ ,$$

and clearly the stochastic integral has zero expectation. Taking expectation on both sides yields

$$\mathbb{E}[X(t)] = \exp\{-\beta\, t\}\ \mathbb{E}[X(0)]\ ,$$

or in other words

$$m(t) = \exp\{-\beta\, t\}\ \mathbb{E}[X(0)]\ .$$

By difference

$$X(t) - m(t) = \exp\{-\beta\, t\}\ (X(0) - m(0)) + \gamma \int_0^t \exp\{-\beta\,(t-s)\}\ dB(s)\ ,$$

hence

$$\begin{aligned}|X(t) - m(t)|^2 \ =\ & \exp\{-2\,\beta\, t\}\ |X(0) - m(0)|^2 + \gamma^2\ |\int_0^t \exp\{-\beta\,(t-s)\}\ dB(s)|^2 \\ & + 2\,\gamma\ \exp\{-\beta\, t\}\ (X(0) - m(0)) \int_0^t \exp\{-\beta\,(t-s)\}\ dB(s)\ ,\end{aligned}$$

Taking expectation on both sides and using the Itô isometry yields

$$\begin{aligned}\mathbb{E}|X(t)-m(t)|^2 \;&=\; \exp\{-2\,\beta\,t\}\;\mathbb{E}|X(0)-m(0)|^2+\gamma^2\int_0^t \exp\{-2\,\beta\,(t-s)\}\;ds\\ &=\; \exp\{-2\,\beta\,t\}\;\mathbb{E}|X(0)-m(0)|^2+\frac{\gamma^2}{2\,\beta}\;(1-\exp\{-2\,\beta\,t\})\;,\end{aligned}$$

or in other words

$$\sigma^2(t)=\exp\{-2\,\beta\,t\}\;\mathbb{E}|X(0)-m(0)|^2+\frac{\gamma^2}{2\,\beta}\;(1-\exp\{-2\,\beta\,t\})\;.$$

By difference

$$X(t+h)-m(t+h)=\exp\{-\beta\,h\}\;(X(t)-m(t))+\gamma\int_t^{t+h}\exp\{-\beta\,(t+h-s)\}\;dB(s)\;,$$

hence

$$\begin{aligned}(X(t+h)-m(t+h))\,(X(t)-m(t))&=\exp\{-2\,\beta\,h\}\;|X(t)-m(t)|^2\\ &+2\,\gamma\,\exp\{-\beta\,h\}\;(X(t)-m(t))\int_t^{t+h}\exp\{-\beta\,(t+h-s)\}\;dB(s)\;.\end{aligned}$$

Taking expectation on both sides yields

$$\mathbb{E}[\,(X(t+h)-m(t+h))\,(X(t)-m(t))\,]\;=\;\exp\{-2\,\beta\,h\}\;\mathbb{E}|X(t)-m(t)|^2\;,$$

or in other words

$$\rho(t,h)=\exp\{-2\,\beta\,h\}\;\sigma^2(t)\;.$$

If the variance does not depend on time (then necessarily the correlation coefficient also does not depend on time), i.e. if $\sigma^2(t)\equiv\sigma^2$ for any $t\geq 0$, then

$$\sigma^2=\exp\{-2\,\beta\,t\}\;\sigma^2+\frac{\gamma^2}{2\,\beta}\;(1-\exp\{-2\,\beta\,t\})\qquad\text{i.e.}\qquad\sigma^2=\frac{\gamma^2}{2\,\beta}\;.$$

□

(iv) **Assume further that the initial condition is a Gaussian random variable. Show that the process $X(t)$ is Gaussian.**

Exercise 6 [Kramers–Smoluchowski approximation] Let $B(t)$ be a one–dimensional standard Brownian motion, let the real–valued drift function $b(x)$ be globally Lipschitz continuous. Consider the one–dimensional SDE

$$Y(t)=Y(0)+\int_0^t b(Y(s))\,ds+B(t)\;,\qquad(\star)$$

with initial condition $Y(0)$ independent of the Brownian motion, and for any positive $\alpha > 0$, consider the two–dimensional SDE

$$\begin{aligned} X(t) &= X(0) + \int_0^t V(s)\,ds \\ V(t) &= \alpha\,[-\int_0^t V(s)\,ds + \int_0^t b(X(s))\,ds + B(t)\,] \end{aligned} \qquad (\star\star)$$

with initial condition $(X(0), V(0)) = (Y(0), 0)$ independent of the Brownian motion. The objective is to show that the (first component of the) solution of $(\star\star)$ provides a smooth (differentiable) approximation of the solution of $(\star)$ uniformly on $[0, T]$, as $\alpha \to \infty$.

(i) **Show that the drift function satisfies the linear growth condition.**

(ii) **Check that there exist a unique solution to the SDE $(\star)$, and a unique solution to the SDE $(\star\star)$.**

(iii) **Show that**

$$|X(t) - Y(t)| \leq \frac{1}{\alpha}\,|V(t)| + L \int_0^t |X(s) - Y(s)|\,ds\ ,$$

for some positive constant $L > 0$.

[Hint: check that

$$X(t) = X(0) - \frac{1}{\alpha}\,V(t) + \int_0^t b(X(s))\,ds + B(t)\ .]$$

Solution

Extracting

$$\int_0^t V(s)\,ds = -\frac{1}{\alpha}\,V(t) + \int_0^t b(X(s))\,ds + B(t)\ ,$$

from the second component of $(\star\star)$ and reporting this expression in the first component of $(\star\star)$ yields

$$X(t) = X(0) - \frac{1}{\alpha}\,V(t) + \int_0^t b(X(s))\,ds + B(t)\ .$$

By difference, and since $X(0) = Y(0)$, it holds

$$X(t) - Y(t) = -\frac{1}{\alpha}\,V(t) + \int_0^t [b(X(s)) - b(Y(s))]\,ds\ ,$$

and using the triangle inequality and the global Lipschitz property yields

$$|X(t) - Y(t)| \leq \frac{1}{\alpha}\,|V(t)| + L \int_0^t |X(s) - Y(s)|\,ds\ .$$

□

(iv) **Show that**

$$\frac{1}{\alpha}\,V(t) = \int_0^t \exp\{-\alpha\,(t-s)\}\,b(X(s))\,ds + B(t) - \alpha\,\int_0^t \exp\{-\alpha\,(t-s)\}\,B(s)\,ds\ .$$

[Hint: use the variation of the constant method.]

Solution

Introducing the process Z defined by $Z(t) = \dfrac{1}{\alpha}\,V(t) - B(t)$ for any $t \geq 0$, it follows from the second component of ($\star\star$) that

$$\begin{aligned} Z(t) &= -\int_0^t V(s)\,ds + \int_0^t b(X(s))\,ds \\ &= -\alpha\,\int_0^t Z(s)\,ds + \int_0^t b(X(s))\,ds - \alpha\,\int_0^t B(s)\,ds\ . \end{aligned}$$

In other words, the process Z satisfies the ODE

$$\frac{d}{dt}Z(t) = -\alpha\,Z(t) + b(X(t)) - \alpha\,B(t)\ .$$

The variation of the constant formula provides an explicit expression for the solution

$$Z(t) = \int_0^t \exp\{-\alpha\,(t-s)\}\ b(X(s))\,ds - \alpha\,\int_0^t \exp\{-\alpha\,(t-s)\}\ B(s)\,ds\ ,$$

hence

$$\frac{1}{\alpha}\,V(t) = \int_0^t \exp\{-\alpha\,(t-s)\}\ b(X(s))\,ds + B(t) - \alpha\,\int_0^t \exp\{-\alpha\,(t-s)\}\ B(s)\,ds\ ,$$

which proves the claim. Note that integration by parts yields

$$B(t)\ \exp\{\alpha\,t\} = \alpha\,\int_0^t \exp\{\alpha\,s\}\ B(s)\,ds + \int_0^t \exp\{\alpha\,s\}\,dB(s)\ ,$$

multiplying both sides by $\exp\{-\alpha\,t\}$ yields

$$B(t) - \alpha\,\int_0^t \exp\{-\alpha\,(t-s)\}\ B(s)\,ds = \int_0^t \exp\{-\alpha\,(t-s)\}\,dB(s)\ ,$$

and reporting this expression above yields

$$\frac{1}{\alpha}\,V(t) = \int_0^t \exp\{-\alpha\,(t-s)\}\ b(X(s))\,ds + \int_0^t \exp\{-\alpha\,(t-s)\}\,dB(s)\ .$$

□

(v) **Show that almost surely**

$$\sup_{0\leq t\leq T}\ |\,B(t) - \alpha\,\int_0^t \exp\{-\alpha\,(t-s)\}\,B(s)\,ds\,| \to 0\ ,$$

as $\alpha \to \infty$**.**

Note that

$$\begin{aligned}
&B(t) - \alpha \int_0^t \exp\{-\alpha\,(t-s)\}\, B(s)\, ds \\
&\quad = -\alpha \int_0^{t-\delta} \exp\{-\alpha\,(t-s)\}\, B(s)\, ds \\
&\qquad + \alpha \int_{t-\delta}^t \exp\{-\alpha\,(t-s)\}\, (B(t) - B(s))\, ds + \exp\{-\alpha\,\delta\}\, B(t)\ ,
\end{aligned}$$

and using the triangle inequality yields

$$\begin{aligned}
&|\, B(t) - \alpha \int_0^t \exp\{-\alpha\,(t-s)\}\, B(s)\, ds\, | \\
&\quad \leq \sup_{0\leq s\leq t} |B(s)|\ \alpha \int_0^{t-\delta} \exp\{-\alpha\,(t-s)\}\, ds \\
&\qquad + \sup_{t-\delta\leq s\leq t} |B(t) - B(s)|\ \alpha \int_{t-\delta}^t \exp\{-\alpha\,(t-s)\}\, ds + \exp\{-\alpha\,\delta\}\, |B(t)| \\
&\quad \leq \sup_{0\leq s\leq t} |B(s)|\ [\exp\{-\alpha\,\delta\} - \exp\{-\alpha\,t\}] \\
&\qquad + \sup_{0\leq u\leq \delta} |B(t) - B(t-u)|\ [1 - \exp\{-\alpha\,\delta\}] + \exp\{-\alpha\,\delta\}\, |B(t)| \\
&\quad \leq \sup_{0\leq s\leq t} |B(s)|\ \exp\{-\alpha\,\delta\} + \sup_{0\leq u\leq \delta} |B(t) - B(t-u)| + \exp\{-\alpha\,\delta\}\, |B(t)|\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{0\leq t\leq T} |\, B(t) - \alpha \int_0^t \exp\{-\alpha\,(t-s)\}\, B(s)\, ds\, | \\
&\quad \leq \sup_{0\leq t\leq T} |B(t)|\ \exp\{-\alpha\,\delta\} + \sup_{\substack{0\leq u,v\leq T \\ |u-v|\leq\delta}} |B(u) - B(v)| + \exp\{-\alpha\,\delta\} \sup_{0\leq t\leq T} |B(t)|\ ,
\end{aligned}$$

and

$$\limsup_{\alpha\to\infty} \sup_{0\leq t\leq T} |\, B(t) - \alpha \int_0^t \exp\{-\alpha\,(t-s)\}\, B(s)\, ds\, | \leq \sup_{\substack{0\leq u,v\leq T \\ |u-v|\leq\delta}} |B(u) - B(v)|\ ,$$

and the right–hand side can be made arbitrary small by taking $\delta > 0$ small enough.

□

(vi) **Show that almost surely**

$$\sup_{0\le t\le T} \frac{1}{\alpha}\,|V(t)| \to 0 \qquad \textbf{hence} \qquad \sup_{0\le t\le T} |X(t)-Y(t)| \to 0\ ,$$

as $\alpha \to \infty$.

Solution

Under the assumptions

$$\sup_{0\le s\le t} |b(X(s))| \le K\,(1+\sup_{0\le s\le t}|X(s)|)\ ,$$

and

$$\begin{aligned}
|\int_0^t \exp\{-\alpha\,(t-s)\}\, b(X(s))\,ds| &\le \int_0^t \exp\{-\alpha\,(t-s)\}\, |b(X(s))|\,ds \\
&\le \sup_{0\le s\le t} |b(X(s))| \int_0^t \exp\{-\alpha\,(t-s)\}\,ds \\
&\le K\,(1+\sup_{0\le s\le t}|X(s)|)\, \frac{1-\exp\{-\alpha\,t\}}{\alpha}\ ,
\end{aligned}$$

hence

$$\sup_{0\le t\le T} |\int_0^t \exp\{-\alpha\,(t-s)\}\, b(X(s))\,ds| \le \frac{1}{\alpha}\,K\,(1+\sup_{0\le t\le T}|X(t)|)\ .$$

Recalling the identity stated in question (iv) and using the triangle inequality yields

$$\begin{aligned}
|\frac{1}{\alpha}\,V(t)| &\le |\int_0^t \exp\{-\alpha\,(t-s)\}\, b(X(s))\,ds| \\
&\qquad + |B(t) - \alpha \int_0^t \exp\{-\alpha\,(t-s)\}\, B(s)\,ds| \\
&\le \sup_{0\le t\le T} |\int_0^t \exp\{-\alpha\,(t-s)\}\, b(X(s))\,ds| \\
&\qquad + \sup_{0\le t\le T} |B(t) - \alpha \int_0^t \exp\{-\alpha\,(t-s)\}\, B(s)\,ds|\ ,
\end{aligned}$$

and using the result proved in question (v) yields

$$\begin{aligned}
\sup_{0\le t\le T} |\frac{1}{\alpha}\,V(t)| &\le \frac{1}{\alpha}\,K\,(1+\sup_{0\le t\le T}|X(t)|) \\
&\qquad + \sup_{0\le t\le T} |B(t) - \alpha \int_0^t \exp\{-\alpha\,(t-s)\}\, B(s)\,ds| \to 0\ ,
\end{aligned}$$

almost surely as $\alpha \uparrow \infty$.

Finally, the bound stated in question (iii) yields

$$|X(t)-Y(t)| \le [\sup_{0\le t\le T} \frac{1}{\alpha}\,|V(t)|\,] + L\int_0^t |X(s)-Y(s)|\,ds\ ,$$

and using the Gronwall lemma yields

$$|X(t)-Y(t)| \le [\sup_{0\le t\le T} \frac{1}{\alpha}\,|V(t)|\,]\ \exp\{L\,t\}$$

and

$$\sup_{0\le t\le T}|X(t)-Y(t)| \le [\sup_{0\le t\le T} \frac{1}{\alpha}\,|V(t)|\,]\ \exp\{L\,T\} \to 0\ ,$$

almost surely as $\alpha \uparrow \infty$.

□

Exercise 7 [SDE for the Brownian bridge] Consider the process defined by

$$Z'(t) = (1-t)\int_0^t \frac{dB(s)}{1-s}\ ,$$

for any $0 \le t < 1$.

(i) **Show that $Z'(t) \to 0$ in L^2 as $t \to 1$ (and define $Z'(1) = 0$ by continuity, assuming that the convergence holds also almost surely). Show that Z' has the same distribution as the Brownian bridge.**

SOLUTION

It follows from the Itô isometry that

$$\mathbb{E}|\int_0^t \frac{dB(s)}{1-s}|^2 = \int_0^t \frac{ds}{(1-s)^2} = \frac{1}{1-t} - 1 = \frac{t}{1-t}\ ,$$

hence

$$\mathbb{E}|Z'(t)|^2 = (1-t)^2\,\mathbb{E}|\int_0^t \frac{dB(s)}{1-s}|^2 = (1-t)^2\,\frac{t}{1-t} = t\,(1-t) \to 0\ ,$$

as $t \to 1$.

It follows again from the Itô isometry that

$$\begin{aligned}
\mathbb{E}[\int_0^t \frac{dB(u)}{1-u}\int_0^s \frac{dB(v)}{1-v}\,] &= \mathbb{E}[\int_0^{\max(t,s)} \frac{dB(u)}{1-u}\int_0^{\min(t,s)} \frac{dB(v)}{1-v}\,] \\
&= \mathbb{E}|\int_0^{\min(t,s)} \frac{dB(u)}{1-u}|^2 \\
&= \int_0^{\min(t,s)} \frac{du}{(1-u)^2} \\
&= \frac{\min(t,s)}{1-\min(t,s)}\ ,
\end{aligned}$$

hence

$$\begin{aligned} K'(t,s) &= \mathbb{E}[Z'(t)\, Z'(s)] \\ &= (1-t)\,(1-s)\;\mathbb{E}[\int_0^t \frac{dB(u)}{1-u}\;\int_0^s \frac{dB(v)}{1-v}\,] \\ &= (1-\min(t,s))\,(1-\max(t,s))\,\frac{\min(t,s)}{1-\min(t,s)} \\ &= \min(t,s)\,(1-\max(t,s))\;. \end{aligned}$$

NOTE ON THE WIENER INTEGRAL

Note that a Wiener integral, i.e. the stochastic integral of a deterministic square–integrable function ϕ, is a Gaussian random variable. Indeed, for any convergent subdivision $0 = t_0 < t_1 < \cdots < t_n = t$, the finite sum

$$\sum_{i=1}^{n} \phi(t_{i-1})\,(B(t_i) - B(t_{i-1}))\;,$$

is a Gaussian random variable, as a linear combination of independant Gaussian random variables, and so is its limit

$$W(t) = \int_0^t \phi(s)\, dB(s)\;.$$

For any integer $n \geq 1$ and any time instants $0 < t_1 < \cdots < t_n < t$, the vector $(W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1}))$ is a Gaussian random vector as a collection of independent Gaussian random variables, hence the vector $(W(t_1), \cdots, W(t_n))$ is a Gaussian random vector as a linear transformation of the Gaussian random vector $(W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1}))$. This shows that the whole process W is Gaussian.

(END OF THE NOTE)

The process Z', defined in terms of a Wiener integral, is Gaussian, and its covariance function coincides with the covariance function of the Brownian bridge Z. Therefore, the two processes Z and Z' have the same finite–dimensional distributions, hence they have the same distribution.

□

(ii) **Show that Z' is the unique solution of the SDE**

$$Z'(t) = -\int_0^t \frac{Z'(s)}{1-s}\, ds + B(t)\;,$$

for any $0 \leq t < 1$.

[Hint: write the Itô formula for Z' seen as the product of two Itô processes.]

SOLUTION

Introducing the two processes

$$u(t) = 1 - t \qquad \text{and} \qquad X(t) = \int_0^t \frac{dB(s)}{1-s} \ ,$$

so that $Z'(t) = u(t)\, X(t)$, and writing the Itô formula for the two–dimensional Itô process

$$\begin{pmatrix} u(t) \\ X(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -1 \\ 0 \end{pmatrix} ds + \int_0^t \begin{pmatrix} 0 \\ \dfrac{1}{1-s} \end{pmatrix} dB(s) \ ,$$

and for the function $f(u,x) = u\,x$, with

$$f'(u,x) = (x \quad u) \qquad \text{and} \qquad f''(u,x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ,$$

yields

$$\begin{aligned} u(t)\, X(t) \ &= \ \int_0^t (X(s) \quad u(s)) \ [\begin{pmatrix} -1 \\ 0 \end{pmatrix} ds + \begin{pmatrix} 0 \\ \dfrac{1}{1-s} \end{pmatrix} dB(s)] \\ &\quad + \tfrac{1}{2} \int_0^t \text{trace}[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{1}{1-s} \end{pmatrix} \begin{pmatrix} 0 & \dfrac{1}{1-s} \end{pmatrix}] \ ds \\ &= \ - \int_0^t X(s)\, ds + \int_0^t \frac{u(s)}{1-s}\, dB(s) \ . \end{aligned}$$

Indeed

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ * \end{pmatrix} \begin{pmatrix} 0 & * \end{pmatrix} = \begin{pmatrix} * \\ 0 \end{pmatrix} \begin{pmatrix} 0 & * \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \ ,$$

hence

$$\text{trace}[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ * \end{pmatrix} \begin{pmatrix} 0 & * \end{pmatrix}] = 0 \ .$$

In other words

$$Z'(t) = - \int_0^t \frac{Z'(s)}{1-s}\, ds + B(t) \ .$$

□