INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems
Introduction to SDE's

TD 2 : Some applications of the Itô formula

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Exercise 1 [Integration by parts] Let $B(t)$ be a d–dimensional standard Brownian motion, and let $X_1(t)$ and $X_2(t)$ be two one–dimensional Itô processes, with

$$X_i(t) = X_i(0) + \int_0^t \psi_i(s)\, ds + \int_0^t \phi_i(s)\, dB(s) \qquad \text{for } i = 1, 2.$$

Here $\phi_1(s)$ and $\phi_2(s)$ are two $1 \times d$ matrices (row vectors).

(i) **Write the Itô formula for the one–dimensional process $X_1(t)\, X_2(t)$.**

Solution

The two–dimensional process $X(t) = (X_1(t), X_2(t))$ is an Itô process, with

$$\psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} \qquad \text{and} \qquad \phi(s) = \begin{pmatrix} \phi_1(s) \\ \phi_2(s) \end{pmatrix}$$

The Itô formula for the Itô process $X(t)$ and for the function $f(x_1, x_2) = x_1\, x_2$, with

$$f'(x_1, x_2) = (x_2 \quad x_1) \qquad \text{and} \qquad f''(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

yields

$$\begin{aligned} X_1(t)\, X_2(t) \;&=\; X_1(0)\, X_2(0) + \int_0^t (X_2(s) \quad X_1(s))\, [\begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} ds + \begin{pmatrix} \phi_1(s) \\ \phi_2(s) \end{pmatrix} dB(s)] \\ &\quad + \tfrac{1}{2} \int_0^t \text{trace}[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(s)\, \phi_1^*(s) & \phi_1(s)\, \phi_2^*(s) \\ \phi_2(s)\, \phi_1^*(s) & \phi_2(s)\, \phi_2^*(s) \end{pmatrix}]\, ds \\ &=\; X_1(0)\, X_2(0) + \int_0^t (X_2(s)\, \psi_1(s) + X_1(s)\, \psi_2(s))\, ds \\ &\quad + \int_0^t (X_2(s)\, \phi_1(s) + X_1(s)\, \phi_2(s))\, dB(s) \\ &\quad + \tfrac{1}{2} \int_0^t (\phi_1(s)\, \phi_2^*(s) + \phi_2(s)\, \phi_1^*(s))\, ds\ , \end{aligned}$$

or in other words

$$X_1(t)\,X_2(t) \;=\; X_1(0)\,X_2(0) + \int_0^t X_2(s)\;dX_1(s) + \int_0^t X_1(s)\;dX_2(s)$$

$$+\,\tfrac{1}{2}\int_0^t (\phi_1(s)\,\phi_2^*(s) + \phi_2(s)\,\phi_1^*(s))\,ds\;.$$

□

Multi–dimensional version: Let $B(t)$ be a d–dimensional standard Brownian motion, and let $X_1(t)$ and $X_2(t)$ be two m–dimensional Itô processes, with

$$X_i(t) = X_i(0) + \int_0^t \psi_i(s)\,ds + \int_0^t \phi_i(s)\,dB(s) \qquad \text{for } i = 1, 2.$$

Here $\psi_1(s)$ and $\psi_2(s)$ are two m–dimensional vectors, and $\phi_1(s)$ and $\phi_2(s)$ are two $m\times d$ matrices.

(ii) **Write the Itô formula for the one–dimensional process $X_1^*(t)\,X_2(t)$ and for the $m \times m$ matrix–valued process $X_1(t)\,X_2^*(t)$.**

[Hint: for the second part, apply the result obtained at question (i) to the two one–dimensional Itô processes $u_1^*\,X_1(t)$ and $u_2^*\,X_2(t)$ where u_1 and u_2 are two arbitrary vectors in $\mathbb{R}^m$.]

Solution

First part (scalar product): The $2m$–dimensional process $X(t) = (X_1(t), X_2(t))$ is an Itô process, with

$$\psi(s) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} \qquad \text{and} \qquad \phi(s) = \begin{pmatrix} \phi_1(s) \\ \phi_2(s) \end{pmatrix}$$

The Itô formula for the Itô process $X(t)$ and for the function $f(x_1, x_2) = x_1^*\,x_2$, with

$$f'(x_1, x_2) = (x_2^* \quad x_1^*) \qquad \text{and} \qquad f''(x_1, x_2) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\;,$$

yields

$$X_1(t)\,X_2^*(t) \;=\; X_1^*(0)\,X_2(0) + \int_0^t (X_2^*(s) \quad X_1^*(s))\,[\begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix}\,ds + \begin{pmatrix} \phi_1(s) \\ \phi_2(s) \end{pmatrix}\,dB(s)]$$

$$+\,\tfrac{1}{2}\int_0^t \text{trace}[\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \phi_1(s)\,\phi_1^*(s) & \phi_1(s)\,\phi_2^*(s) \\ \phi_2(s)\,\phi_1^*(s) & \phi_2(s)\,\phi_2^*(s) \end{pmatrix}]\,ds$$

$$=\; X_1^*(0)\,X_2(0) + \int_0^t (X_2^*(s)\,\psi_1(s) + X_1^*(s)\,\psi_2(s))\,ds$$

$$+\int_0^t (X_2^*(s)\,\phi_1(s) + X_1^*(s)\,\phi_2(s))\,dB(s)$$

$$+\,\tfrac{1}{2}\int_0^t \text{trace}[\phi_1(s)\,\phi_2^*(s) + \phi_2(s)\,\phi_1^*(s)]\,ds\;,$$

or in other words

$$X_1^*(t)\, X_2(t) \;=\; X_1^*(0)\, X_2(0) + \int_0^t X_2^*(s)\; dX_1(s) + \int_0^t X_1^*(s)\; dX_2(s)$$
$$+ \tfrac{1}{2} \int_0^t \operatorname{trace}[\phi_1(s)\, \phi_2^*(s) + \phi_2(s)\, \phi_1^*(s)]\, ds\ .$$

Second part: If u_1 and u_2 are m–dimensional vectors, then $u_1^*\, X_1(t)$ and $u_2^*\, X_2(t)$ are two one–dimensional Itô processes, with

$$u_i^*\, X_i(t) = u_i^*\, X_i(0) + \int_0^t u_i^*\, \psi_i(s)\, ds + \int_0^t u_i^*\, \phi_i(s)\, dB(s) \qquad \text{for } i = 1, 2.$$

Here $u_1^*\, \phi_1(s)$ and $u_1^*\, \phi_2(s)$ are two $1 \times d$ matrices (row vectors). Applying the result obtained at question (i) yields

$$u_1^*\, X_1(t)\, u_2^*\, X_2(t) \;=\; u_1^*\, X_1(0)\, u_2^*\, X_2(0) + \int_0^t u_2^*\, X_2(s)\; d[u_1^*\, X_1(s)] + \int_0^t u_1^*\, X_1(s)\; d[u_2^*\, X_2(s)]$$
$$+ \tfrac{1}{2} \int_0^t [u_1^*\, \phi_1(s)\, u_2^*\, \phi_2^*(s) + u_2^*\, \phi_2(s)\, u_1^*\, \phi_1^*(s)]\, ds\ ,$$

or equivalently, after rearranging terms

$$u_1^*\, X_1(t)\, X_2^*(t)\, u_2 \;=\; u_1^*\, X_1(0)\, X_2^*(0)\, u_2 + \int_0^t u_1^*\, dX_1(s)\, X_2^*(s)\, u_2 + \int_0^t u_1^*\, X_1(s)\; dX_2^*(s)\, u_2$$
$$+ \int_0^t u_1^*\, \phi_1(s)\, \phi_2^*(s)\, u_2\, ds\ ,$$

or in other words

$$X_1(t)\, X_2^*(t) \;=\; X_1(0)\, X_2^*(0) + \int_0^t dX_1(s)\, X_2^*(s) + \int_0^t X_1(s)\; dX_2^*(s)$$
$$+ \int_0^t \phi_1(s)\, \phi_2^*(s)\, ds\ ,$$

since the vectors u_1 and u_2 are arbitrary.

□

Problem 2 [Burkholder–Davis–Gundy inequalities] Let B be a one–dimensional standard Brownian motion, and for any $\phi \in M^2([0, T])$ define

$$M(t) = \int_0^t \phi(s)\, dB(s)\ , \qquad M_*(t) = \max_{0 \le s \le t} |M(s)|\ , \qquad A(t) = \int_0^t |\phi(s)|^2\, ds\ .$$

The objective is to show that for any $p \ge 2$, there exist positive constants $0 < c_p \le C_p < \infty$ such that for any $0 \le t \le T$

$$c_p\ \mathbb{E}|A(t)|^{p/2} \le \mathbb{E}|M_*(t)|^p \le C_p\ \mathbb{E}|A(t)|^{p/2}\ ,$$

i.e. the moments of a martingale can be controlled in terms of the moments of its increasing process.

(i) **Show that the upper bound holds for $p = 2$.**

[Hint: use the Doob inequality.]

Solution

The Doob inequality for $p = 2$ provides a uniform control in terms of the terminal value

$$\mathbb{E}[\max_{0\le s\le t}|M(s)|^2] \le 4\,\mathbb{E}|M(t)|^2 \ ,$$

and the Itô isometry yields

$$\mathbb{E}|M(t)|^2 = \mathbb{E}\int_0^t |\phi(s)|^2\,ds = \mathbb{E}[A(t)] \ .$$

Combining the two estimates provide a uniform control in terms of the increasing process, i.e.

$$\mathbb{E}[\max_{0\le s\le t}|M(s)|^2] \le 4\,\mathbb{E}[A(t)] \ .$$

□

The following boundedness assumption will be used:

there exists a positive $K > 0$ such that $A(T) \le K^2$ and $|M(t)| \le K$ for any $0 \le t \le T$.

(ii) **Assume that the result holds under the boundedness assumption. Show that the result can be extended to the general case.**

[Hint: for any $n \ge 1$, consider the stopping time

$$\tau_n = \inf\{0 \le t \le T \,:\, |M(t)| \ge n \text{ or } A(t) \ge n\} \qquad \text{or} \qquad \tau_n = T \ ,$$

and the stopped martingale defined by $M^n(t) = M(t \wedge \tau_n)$ for any $0 \le t \le T$.]

From now on, the boundedness assumption is made.

Upper bound:

(iii) **Show that the p–th order moment $\mathbb{E}|M_*(t)|^p$ can be bounded in terms of $\mathbb{E}|M(t)|^p$.**

[Hint: use the Doob inequality.]

Solution

It follows from the Doob inequality that

$$\{\mathbb{E}[\max_{0\le s\le t}|M(s)|^p\,]\}^{1/p} \le \frac{p}{p-1}\,\{\mathbb{E}|M(t)|^p\}^{1/p} \ ,$$

or in other words

$$\mathbb{E}|M_*(t)|^p = \mathbb{E}[\max_{0\le s\le t}|M(s)|^p\,] \le (\frac{p}{p-1})^p\,\mathbb{E}|M(t)|^p \ .$$

□

(iv) **Write the Itô formula for $|M(t)|^p$. Show that the p–th order moment $\mathbb{E}|M(t)|^p$ can be bounded in terms of $\mathbb{E}[|M_*(t)|^{p-2}\, A(t)]$, and further bounded in terms of $\mathbb{E}|M_*(t)|^p$ and $\mathbb{E}|A(t)|^{p/2}$.**

[Hint: for the last part, use the Hölder inequality.]

Solution

The Itô formula for the Itô process

$$M(t) = \int_0^t \phi(s)\, dB(s) \ ,$$

and for the function $g(x) = |x|^p$, with

$$f'x) = p\,|x|^{p-1}\,\text{sign}(x) \qquad \text{and} \qquad f''(x) = p\,(p-1)\,|x|^{p-2} \ ,$$

yields

$$|M(t)|^p = p \int_0^t |M(s)|^{p-1}\,\text{sign}(M(s))\,\phi(s)\,dB(s) + \tfrac{1}{2}\,p\,(p-1) \int_0^t |M(s)|^{p-2}\,|\phi(s)|^2\,ds \ .$$

Under the boundedness assumption, the integrand $s \mapsto |M(s)|^{p-1}\ \text{sign}(M(s))\,\phi(s)$ belongs to $M^2([0,T])$, since

$$\mathbb{E}\int_0^T ||M(s)|^{p-1}\,\text{sign}(M(s))\,\phi(s)|^2\,ds \leq K^{2p-2}\,\mathbb{E}\int_0^T |\phi(s)|^2\,ds \leq K^{2p} < \infty \ ,$$

and therefore the stochastic integral is a martingale and its expectation is zero. Taking expectation yields

$$\begin{aligned}
\mathbb{E}|M(t)|^p \ &= \ \tfrac{1}{2}\,p\,(p-1)\,\mathbb{E}[\int_0^t |M(s)|^{p-2}\,|\phi(s)|^2\,ds] \\
&\leq \ \tfrac{1}{2}\,p\,(p-1)\,\mathbb{E}[\max_{0\leq s\leq t} |M(s)|^{p-2} \int_0^t |\phi(s)|^2\,ds] \\
&= \ \tfrac{1}{2}\,p\,(p-1)\,\mathbb{E}[\,|M_*(t)|^{p-2}\,A(t)] \ .
\end{aligned}$$

The Hölder inequality for conjugate exponents $q = p/(p-2)$ and $q' = \frac{1}{2}\,p$ yields

$$\begin{aligned}
\mathbb{E}[|M_*(t)|^{p-2}\,A(t)] \ &\leq \ \{\mathbb{E}|M_*(t)|^{q\,(p-2)}\}^{1/q}\,\{\mathbb{E}|A(t)|^{q'}\}^{1/q'} \\
&= \ \{\mathbb{E}|M_*(t)|^p\}^{1-2/p}\,\{\mathbb{E}|A(t)|^{p/2}\}^{2/p} \ ,
\end{aligned}$$

hence

$$\mathbb{E}|M(t)|^p \leq \tfrac{1}{2}\,p\,(p-1)\,\{\mathbb{E}|M_*(t)|^p\}^{1-2/p}\,\{\mathbb{E}|A(t)|^{p/2}\}^{2/p} \ .$$

Reporting this inequality in the inequality obtained in question (iii) yields

$$\begin{aligned}
\mathbb{E}|M_*(t)|^p \ &\leq \ (\frac{p}{p-1})^p\,\mathbb{E}|M(t)|^p \\
&\leq \ (\frac{p}{p-1})^p\,\tfrac{1}{2}\,p\,(p-1)\,\{\mathbb{E}|M_*(t)|^p\}^{1-2/p}\,\{\mathbb{E}|A(t)|^{p/2}\}^{2/p} \ ,
\end{aligned}$$

and collecting all $\mathbb{E}|M_*(t)|^p$ terms in the left–hand side yields

$$\{\mathbb{E}|M_*(t)|^p\}^{2/p} \le (\frac{p}{p-1})^p \ \tfrac{1}{2}\, p\,(p-1)\ \{\mathbb{E}|A(t)|^{p/2}\}^{2/p} \ ,$$

or equivalently

$$\mathbb{E}|M_*(t)|^p \le [\,(\frac{p}{p-1})^p \ \tfrac{1}{2}\, p\,(p-1)\,]^{p/2}\ \mathbb{E}|A(t)|^{p/2} \ .$$

-- □

Lower bound: Define

$$Y(t) = \int_0^t |A(s)|^{(p-1)/2}\ \phi(s)\, dB(s) \ .$$

(v) **Show that** $E|Y(t)|^2 = \frac{1}{p}\ \mathbb{E}|A(t)|^p$**.**

———————— Solution ————————

Under the boundedness assumption, the integrand $s \mapsto |A(s)|^{(p-1)/2}\ \phi(s)$ belongs to $M^2([0,T])$, since

$$\mathbb{E}\int_0^T |A(s)|^{p-1}\ |\phi(s)|^2\, ds \le K^{2p-2}\ \mathbb{E}\int_0^T |\phi(s)|^2\, ds \le K^{2p} < \infty \ ,$$

and the Itô isometry yields

$$\mathbb{E}|Y(t)|^2 = \mathbb{E}\int_0^T |A(s)|^{p-1}\ |\phi(s)|^2\, ds \ .$$

On the other hand, the usual chain rule yields

$$\frac{d}{dt}|A(t)|^p = p\ |A(t)|^{p-1}\ |\phi(t)|^2 \ ,$$

or, in integrated form

$$|A(t)|^p = p\ \int_0^t |A(s)|^{p-1}\ |\phi(s)|^2\, ds \ .$$

Taking expectation yields

$$\mathbb{E}|A(t)|^p = p\,\mathbb{E}\int_0^t |A(t)|^{p-1}\ |\phi(s)|^2\, ds = p\,\mathbb{E}|Y(t)|^2 \ . \qquad \square$$

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(vi) **Using the Itô formula for** $M(t)\,|A(t)|^{(p-1)/2}$**, get an alternate expression for** $Y(t)$ **and show the bound** $|Y(t)| \le 2\,M_*(t)\,|A(t)|^{(p-1)/2}$**. Show that the** p**–th order moment** $\mathbb{E}|A(t)|^p$ **can be bounded in terms of** $\mathbb{E}|M_*(t)|^{2p}$ **and** $\mathbb{E}|A(t)|^p$**.**

[Hint: for the last part, use the Hölder inequality.]

Solution

Note that the usual chain rule yields

$$\frac{d}{dt}|A(t)|^{(p-1)/2} = \tfrac{1}{2}\,(p-1)\;|A(t)|^{(p-3)/2}\;|\phi(t)|^2\;,$$

or, in integrated form

$$|A(t)|^{(p-1)/2} = \tfrac{1}{2}\,(p-1)\;\int_0^t |A(s)|^{(p-3)/2}\;|\phi(s)|^2\,ds\;.$$

The Itô formula (integration by parts) for the two component Itô process

$$\begin{pmatrix} M(t) \\ |A(t)|^{(p-1)/2} \end{pmatrix} = \tfrac{1}{2}\,(p-1)\;\int_0^t \begin{pmatrix} 0 \\ |A(s)|^{(p-3)/2}\;|\phi(s)|^2 \end{pmatrix}\;ds + \int_0^t \begin{pmatrix} \phi(s) \\ 0 \end{pmatrix}\;dB(s)\;,$$

yields

$$\begin{aligned} M(t)\,|A(t)|^{(p-1)/2} \;&=\; \tfrac{1}{2}\,(p-1)\;\int_0^t M(s)\;|A(s)|^{(p-3)/2}\;|\phi(s)|^2\,ds \\ &\quad + \int_0^t |A(s)|^{(p-1)/2}\;\phi(s)\,dB(s)\;, \end{aligned}$$

hence

$$\begin{aligned} Y(t) \;&=\; \int_0^t |A(s)|^{(p-1)/2}\;\phi(s)\,dB(s) \\ &=\; M(t)\,|A(t)|^{(p-1)/2} - \tfrac{1}{2}\,(p-1)\;\int_0^t M(s)\;|A(s)|^{(p-3)/2}\;|\phi(s)|^2\,ds\;. \end{aligned}$$

Note that

$$|M(t)\,|A(t)|^{(p-1)/2} \le M_*(t)\;|A(t)|^{(p-1)/2}\;,$$

and

$$\begin{aligned} |\tfrac{1}{2}\,(p-1)\;\int_0^t M(s)\;|A(s)|^{(p-3)/2}\;|\phi(s)|^2\,ds| \;&\le\; M_*(t)\;\tfrac{1}{2}\,(p-1)\;\int_0^t |A(s)|^{(p-3)/2}\;|\phi(s)|^2\,ds \\ &\le\; M_*(t)\;|A(t)|^{(p-1)/2}\;. \end{aligned}$$

and the triangle inequality yields

$$|Y(t)| \le 2\,M_*(t)\;|A(t)|^{(p-1)/2}\;.$$

Therefore

$$\mathbb{E}|A(t)|^p \;=\; p\,\mathbb{E}|Y(t)|^2 \le 4\,p\,\mathbb{E}[|M_*(t)|^2\;|A(t)|^{p-1}]\;.$$

The Hölder inequality for conjugate exponents $q = p/(p-1)$ and $q' = p$ yields

$$\begin{aligned}\mathbb{E}[|M_*(t)|^2 \, |A(t)|^{p-1}] \; &\leq \; \{\mathbb{E}|M_*(t)|^{2q'})\}^{1/q'} \, \{\mathbb{E}|A(t)|^{q\,(p-1)}\}^{1/q} \\ &= \; \{\mathbb{E}|M_*(t)|^{2p}\}^{1/p} \, \{\mathbb{E}|A(t)|^{p}\}^{1-1/p} \ ,\end{aligned}$$

hence

$$\mathbb{E}|A(t)|^p \leq 4\,p\, \{\mathbb{E}|M_*(t)|^{2p}\}^{1/p} \, \{\mathbb{E}|A(t)|^{p}\}^{1-1/p} \ ,$$

and collecting all $\mathbb{E}|A(t)|^p$ terms in the left–hand side yields

$$\{\mathbb{E}|A(t)|^p\}^{1/p} \leq 4\,p\, \{\mathbb{E}|M_*(t)|^{2p}\}^{1/p} \ ,$$

or equivalently

$$\mathbb{E}|A(t)|^p \leq (4\,p)^p \, \mathbb{E}|M_*(t)|^{2p} \ .$$

$\square$

Exercice 3 [Exponential bound] Let B be a one–dimensional standard Brownian motion, and for any $\phi \in M^2([0,T])$ define

$$M(t) = \int_0^t \phi(s)\, dB(s) \ , \qquad A(t) = \int_0^t |\phi(s)|^2 \, ds \ .$$

For any positive $\lambda > 0$ define

$$Z^\lambda(t) = \exp\{\lambda\, M(t) - \tfrac{1}{2}\, \lambda^2\, A(t)\} \ .$$

In the special case where $\phi \equiv 1$, the process M is a Brownian motion, and the process Z^λ is a martingale. This was shown using the expression of the Laplace transform of a Gaussian r.v. and this trick cannot be used in the general case.

(i) **Write the Itô formula for $Z^\lambda(t)$, show that it is a (local) martingale, hence $\mathbb{E}[Z^\lambda(t)] \leq 1$ for any $0 \leq t \leq T$.**

[Hint: a nonnegative local martingale is a (true) supermartingale.]

Solution

The Itô formula for the Itô process

$$X(t) = \lambda\, M(t) - \tfrac{1}{2}\, \lambda^2\, A(t) = \int_0^t (-\tfrac{1}{2}\, \lambda^2\, |\phi(s)|^2)\, ds + \int_0^t \lambda\, \phi(s)\, dB(s) \ ,$$

and for the function $f(x) = \exp\{x\}$, with

$$f'(x) = f''(x) = \exp\{x\} \ ,$$

yields

$$
\begin{aligned}
Z^{\lambda}(t) \;&=\; 1 + \int_0^t Z^{\lambda}(s)\, [\, -\tfrac{1}{2}\,\lambda^2\, |\phi(s)|^2\, ds + \lambda\, \phi(s)\, dB(s)] + \tfrac{1}{2} \int_0^t Z^{\lambda}(s)\; \lambda^2\, |\phi(s)|^2\; ds \\
&=\; 1 + \lambda \int_0^t Z^{\lambda}(s)\, \phi(s)\, dB(s)\ .
\end{aligned}
\tag{$\star$}
$$

The integrand $s \mapsto Z^{\lambda}(s)\, \phi(s)$ belongs to M^2_{loc} only, since

$$
\int_0^T |Z^{\lambda}(t)\, \phi(t)|^2\, dt \le \max_{0 \le t \le T} |Z^{\lambda}(t)|^2 \int_0^T |\phi(t)|^2\, dt < \infty
$$

almost surely, for any $T \ge 0$, and therefore the stochastic integral is only a (nonnegative) local martingale, and its expectation is not necessarily zero.

However, a nonnegative local martingale is a supermartingale. Indeed, let L be a nonnegative local martingale, i.e. there exists a non–decreasing sequence of stopping times, such that $\tau_n \uparrow \infty$ almost surely and such that the stopped process defined by $L(t \wedge \tau_n)$ for any $t \ge 0$ is a martingale. Then, for any $0 \le s \le t$ and any $A \in \mathcal{F}(s)$ it holds

$$
\mathbb{E}[1_A\, 1_{(\tau_n\, \ge\, s)}\; L(s)] = \mathbb{E}[1_A\, 1_{(\tau_n\, \ge\, s)}\; L(s \wedge \tau_n)] = \mathbb{E}[1_A\, 1_{(\tau_n\, \ge\, s)}\; L(t \wedge \tau_n)]\ ,
$$

since $A \cap \{\tau_n \ge s\} \in \mathcal{F}(s)$. The Lebesgue dominated convergence theorem yields

$$
\lim_{n \uparrow \infty} \mathbb{E}[1_A\, 1_{(\tau_n\, \ge\, s)}\; L(s)] = \mathbb{E}[1_A\; L(s)]\ ,
$$

and the Fatou lemma yields

$$
\lim_{n \uparrow \infty} \mathbb{E}[1_A\, 1_{(\tau_n\, \ge\, s)}\; L(t \wedge \tau_n)] \ge \mathbb{E}[1_A\; \liminf_{n \uparrow \infty} [1_{(\tau_n\, \ge\, s)}\; L(t \wedge \tau_n)]\,] = \mathbb{E}[1_A\; L(t)]\ ,
$$

hence

$$
\mathbb{E}[1_A\; L(s)] \ge \mathbb{E}[1_A\; L(t)]\ ,
$$

for any $A \in \mathcal{F}(s)$, i.e.

$$
L(s) \ge \mathbb{E}[L(t) \mid \mathcal{F}(s)]\ .
$$

Therefore, the stochastic integral in $(\star)$ is a supermartingale and its expectation is smaller than zero, hence

$$
\mathbb{E}[Z^{\lambda}(t)] \ge 1\ .
$$

-- □

(ii) **Assume that $A(t) \le K\, t$ for any $0 \le t \le T$. Show the following exponential bound: for any positive $c > 0$**

$$
\mathbb{P}[\max_{0 \le t \le T} |M(t)| > c] \le 2\, \exp\{-\frac{c^2}{2\, K\, T}\}\ .
$$

[Hint: use the Chernoff approach to large deviations estimates, and the (easy) inequality

$$\mu\, \mathbb{P}[\max_{0\le t\le T} X(t) > \mu] \le \mathbb{E}[X(0)]\ ,$$

valid for any nonnegative supermartingale.]

Solution

The proof of the inequality for a nonnegative supermartingale follows the same lines as in the discrete times case. Indeed, let L be a nonnegative supermartingale, and introduce the bounded stopping time

$$\tau = \inf\{0 \le s \le t\ :\ L(s) \ge \mu\}\ ,$$

if such time exists, and $\tau = t$ otherwise.

Note that if $\max\limits_{0\le s\le t} L(s) \ge \mu$, then $L(s) \ge \mu$ for some $0 \le s \le t$, hence $L(\tau) \ge \mu$.

It follows from the optional sampling theorem that

$$\begin{aligned}
\mathbb{E}[L(0)] &\ge \mathbb{E}[L(\tau)] \\
&= \mathbb{E}[1_{(\max\limits_{0\le s\le t} L(s) \ge \mu)}\ L(\tau)] + \mathbb{E}[1_{(\max\limits_{0\le s\le t} L(s) < \mu)}\ L(\tau)] \\
&\ge \mu\, \mathbb{P}[\max_{0\le s\le t} L(s) \ge \mu]\ ,
\end{aligned}$$

hence

$$\mu\, \mathbb{P}[\max_{0\le s\le t} L(s) \ge \mu] \le \mathbb{E}[L(0)]\ ,$$

and the claim is proved.

Note that for any $0 \le s \le t$

$$\lambda\, M(s) = \lambda\, M(s) - \tfrac{1}{2}\,\lambda^2\, A(s) + \tfrac{1}{2}\,\lambda^2\, A(s) \le \lambda\, M(s) - \tfrac{1}{2}\,\lambda^2\, A(s) + \tfrac{1}{2}\,\lambda^2\, K\, t\ ,$$

hence for any positive $\lambda > 0$

$$\begin{aligned}
\max_{0\le s\le t} M(s) \ge c\ &\Rightarrow\ \max_{0\le s\le t}[\lambda\, M(s) - \tfrac{1}{2}\,\lambda^2\, A(s)] + \tfrac{1}{2}\,\lambda^2\, K\, t \ge \lambda\, c \\
&\Rightarrow\ \max_{0\le s\le t}[\lambda\, M(s) - \tfrac{1}{2}\,\lambda^2\, A(s)] \ge \lambda\, c - \tfrac{1}{2}\,\lambda^2\, K\, t \\
&\Rightarrow\ \max_{0\le s\le t} Z^\lambda(s) \ge \exp\{\lambda\, c - \tfrac{1}{2}\,\lambda^2\, K\, t\}\ .
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{P}[\max_{0\le s\le t} M(s) \ge c] &\le \mathbb{P}[\max_{0\le s\le t} Z^\lambda(s) \ge \exp\{\lambda\, c - \tfrac{1}{2}\,\lambda^2\, K\, t\}] \\
&\le \exp\{-\lambda\, c + \tfrac{1}{2}\,\lambda^2\, K\, t\}\ ,
\end{aligned}$$

since Z^λ is a nonnegative supermartingale with $Z^\lambda(0) = 1$.

Similarly, note that for any $0 \le s \le t$

$$\lambda\,(-M(s)) = -\lambda\, M(s) - \tfrac{1}{2}\,\lambda^2\, A(s) + \tfrac{1}{2}\,\lambda^2\, A(s) \le -\lambda\, M(s) - \tfrac{1}{2}\,\lambda^2\, A(s) + \tfrac{1}{2}\,\lambda^2\, K\, t\ ,$$

hence for any positive $\lambda > 0$

$$\begin{aligned}\max_{0\le s\le t}(-M(s)) \ge c\ &\Rightarrow\ \max_{0\le s\le t}[-\lambda\, M(s) - \tfrac{1}{2}\,\lambda^2\, A(s)] + \tfrac{1}{2}\,\lambda^2\, K\, t \ge \lambda\, c \\ &\Rightarrow\ \max_{0\le s\le t}[-\lambda\, M(s) - \tfrac{1}{2}\,\lambda^2\, A(s)] \ge \lambda\, c - \tfrac{1}{2}\,\lambda^2\, K\, t \\ &\Rightarrow\ \max_{0\le s\le t} Z^{-\lambda}(s) \ge \exp\{\lambda\, c - \tfrac{1}{2}\,\lambda^2\, K\, t\}\ .\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{P}[\max_{0\le s\le t}(-M(s)) \ge c]\ &\le\ \mathbb{P}[\max_{0\le s\le t} Z^{-\lambda}(s) \ge \exp\{\lambda\, c - \tfrac{1}{2}\,\lambda^2\, K\, t\}] \\ &\le\ \exp\{-\lambda\, c + \tfrac{1}{2}\,\lambda^2\, K\, t\}\ ,\end{aligned}$$

since $Z^{-\lambda}$ is a nonnegative supermartingale with $Z^{-\lambda}(0) = 1$.

Combining the two estimates yields

$$\begin{aligned}\mathbb{P}[\max_{0\le s\le t} |M(s)| \ge c]\ &\le\ \mathbb{P}[\max_{0\le s\le t} M(s) \ge c] + \mathbb{P}[\min_{0\le s\le t} M(s) \le -c] \\ &=\ \mathbb{P}[\max_{0\le s\le t} M(s) \ge c] + \mathbb{P}[\max_{0\le s\le t}(-M(s)) \ge c] \\ &\le\ 2\, \exp\{-\lambda\, c + \tfrac{1}{2}\,\lambda^2\, K\, t\}\ .\end{aligned}$$

The bound holds for any positive $\lambda > 0$, hence it holds also for the minimum over all possible values of $\lambda > 0$. The minimum is achieved for $\lambda = c/(K\, t) > 0$ and the minimum value is $2\, \exp\{-\frac{1}{2}\, c^2/(K\, t)\}$, hence

$$\mathbb{P}[\max_{0\le s\le t} |M(s)| \ge c] \le 2\, \min_{\lambda>0} \exp\{-\lambda\, c + \tfrac{1}{2}\,\lambda^2\, K\} = 2\, \exp\{-\tfrac{1}{2}\,\frac{c^2}{K\, t}\}\ .$$

□

(iii) **In the general case, show that for any positive** $c, K > 0$

$$\mathbb{P}[\max_{0\le t\le T} |M(t)| > c] \le 2\, \exp\{-\frac{c^2}{2\, K\, T}\} + \mathbb{P}[A(T) > K\, T]\ .$$

Solution

Simply

$$\begin{aligned}\mathbb{P}[\max_{0\le t\le T} |M(t)| > c]\ &=\ \mathbb{P}[\max_{0\le t\le T} |M(t)| > c, A(T) \le K\, T] + \mathbb{P}[\max_{0\le t\le T} |M(t)| > c, A(T) > K\, T] \\ &\le\ 2\, \exp\{-\frac{c^2}{2\, K\, T}\} + \mathbb{P}[A(T) > K\, T]\ . \qquad \square\end{aligned}$$

□

Exercise 4 [Feynman–Kac formula] Consider the following linear parabolic PDE

$$\frac{\partial u}{\partial t}(t,x) = \tfrac{1}{2}\,\Delta\, u(t,x) - c(x)\; u(t,x)\;, \qquad \text{for any } (t,x) \in [0,\infty) \times \mathbb{R}^d$$

with initial condition $u(0,x) = g(x)$ for any $x \in \mathbb{R}^d$. Here, the coefficient $c(x)$ is non–negative and bounded from above, its derivative $c'(x)$ is bounded, and the initial condition $g(x)$ together with its derivative $g'(x)$ are bounded. It is assumed that a solution $u(t,x)$ exists that is $C^{1,2}$ with a bounded derivative w.r.t. the space variable.

Fix $x \in \mathbb{R}^d$ and let B be a standard d–dimensional Brownian motion starting from $B(0) = x$.

(i) **Fix $t > 0$, and write the Itô formula for**

$$u(t-s, B(s))\; \exp\{-\int_0^s c(B(r))\, dr\}\;.$$

[Hint: show that the process

$$V(s) = \exp\{-\int_0^s c(B(r))\, dr\}\;,$$

is an Itô process, and write the Itô formula for the $(d+1)$–dimensional Itô process $(B(s), V(s))$ and for the time–dependent function $f(s,x,v) = u(t-s,x)\, v$.]

Solution

The usual chain rule yields

$$\frac{d}{dt} V(t) = -c(B(t))\, V(t)\;,$$

or in integrated form

$$V(s) = 1 - \int_0^s c(B(r)))\, V(r)\, dr\;.$$

Next, the Itô formula for the $(d+1)$–dimensional Itô process

$$\begin{pmatrix} B(s) \\ V(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^s \begin{pmatrix} 0 \\ -c(B(r))\, V(r) \end{pmatrix} dr + \int_0^s \begin{pmatrix} I \\ 0 \end{pmatrix} dB(r)\;,$$

and for the time–dependent function $f(s,x,v) = u(t-s,x)\; v$, with

$$\frac{\partial f}{\partial t}(s,x,v) = -\frac{\partial u}{\partial t}(t-s,x)\; v$$

and

$$f'(s,x,v) = (\frac{\partial u}{\partial x}(t-s,x)\; v \quad u(t-s,x)) \quad \text{and} \quad f''(s,x,v) = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x^2}(t-s,x)\; v & \dfrac{\partial u}{\partial x}(t-s,x) \\ \dfrac{\partial u}{\partial x}(t-s,x) & 0 \end{pmatrix}$$

yields

$$
\begin{aligned}
u(t-s,B(s))\,V(s) \;=\;& u(t,x)+\int_0^s [-\frac{\partial u}{\partial t}(t-r,B(r))\;V(r)]\,dr \\
&+\int_0^s (\frac{\partial u}{\partial x}(t-r,B(r))\;V(r) \quad u(t-r,B(r)))\;\begin{pmatrix}0\\ -c(B(r))\,V(r)\end{pmatrix}\,dr \\
&+\int_0^s (\frac{\partial u}{\partial x}(t-r,B(r))\;V(r) \quad u(t-r,B(r)))\;\begin{pmatrix}I\\ 0\end{pmatrix}\,dB(r) \\
&+\int_0^s \mathrm{trace}[\begin{pmatrix}\frac{\partial^2 u}{\partial x^2}(t-r,B(r))\;V(r) & \frac{\partial u}{\partial x}(t-r,B(r)) \\ \frac{\partial u}{\partial x}(t-r,B(r)) & 0\end{pmatrix}\;\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}]\,dr \\
=\;& ui(t,x)-\int_0^s \frac{\partial u}{\partial t}(t-r,B(r))\;V(r)\,dr \\
&-\int_0^s c(B(r))\,u(t-r,B(r))\,V(r)\,dr \\
&+\int_0^s \frac{\partial u}{\partial x}(t-r,B(r))\,V(r)\,dB(r) \\
&+\tfrac{1}{2}\int_0^s \Delta\,u(t-r,B(r))\,V(r)\,dr
\end{aligned}
$$

Collecting all the ordinary integral terms reduces to

$$
-\int_0^s [\,\frac{\partial u}{\partial t}(t-r,B(r))-\tfrac{1}{2}\,\Delta\,u(t-r,B(r))+c(B(r))\,u(t-r,B(r))\,]\,V(r)\,dr=0
$$

since

$$
\frac{\partial u}{\partial t}(s,y)-\tfrac{1}{2}\,\Delta\,u(s,y)+c(y)\;u(s,y)=0\;,
$$

for any $s\geq 0$ and any $y\in\mathbb{R}^d$, and the identity holds in particular for $s=t-r$ and for $y=B(r)$. Therefore

$$
u(t-s,B(s))\,V(s)=u(t,x)+\int_0^s \frac{\partial u}{\partial x}(t-r,B(r))\,V(r)\,dB(r)\;,
$$

and in particular for $s=t$ it holds

$$
u(t,x)+\int_0^t \frac{\partial u}{\partial x}(t-r,B(r))\,V(r)\,dB(r)=u(0,B(t))\,V(t)=g(B(t))\;\exp\{-\int_0^t c(B(r))\,dr\}\;.
$$

□

(ii) **Show that**

$$
u(t,x)=\mathbb{E}_{0,x}[g(B(t))\;\exp\{-\int_0^t c(B(r))\,dr\}]\;.
$$

(iii) **Check a posteriori that the derivative w.r.t. the space variable is bounded.**

[Hint: write $B(t) = x + B_0(t)$ with another standard d–dimensional Brownian motion starting from $B_0(0) = 0$.]

Exercise 5 [Wong–Zakai approximation] Let B be a one–dimensional standard Brownian motion, and let

$$B_n(t) = \int_0^t \dot{B}_n(s)\,ds\ ,$$

be an absolutely continuous approximation, such that $B_n(t) \to B(t)$ almost surely as $n \uparrow \infty$. Let f be a twice differentiable function, and let $u = f'$.

(i) **Write the usual chain rule (change of variable formula) for $f(B_n(t))$.**

Solution

The usual chain rule yields

$$\frac{d}{dt} f(B_n(t)) = f'(B_n(t))\,\dot{B}_n(t)\ ,$$

or in integrated form

$$f(B_n(t)) = f(B_n(0)) + \int_0^t f'(B_n(s))\,\dot{B}_n(s)\,ds\ .$$

□

(ii) **Write the Itô formula for $f(B(t))$.**

Solution

The Itô formula yields

$$f(B(t)) = f(B(0)) + \int_0^t f'(B(s))\,dB(s) + \tfrac{1}{2}\int_0^t f''(B(s))\,ds\ .$$

□

(iii) **What is the limit as $n \uparrow \infty$ of**

$$\int_0^t u(B_n(s))\,dB_n(s) = \int_0^t u(B_n(s))\,\dot{B}_n(s)\,ds\ ?$$

Solution

Clearly, $f(B_n(t)) - f(B_n(0)) \to f(B(t)) - f(B(0))$, hence

$$\int_0^t u(B_n(s))\,\dot{B}_n(s)\,ds \to \int_0^t u(B(s))\,dB(s) + \tfrac{1}{2}\int_0^t u'(B(s))\,ds\ ,$$

as $n \uparrow \infty$.

□

As an illustration, one can consider the polygonal approximation

$$B_n(s) = \frac{B(t_{i-1}^n)\,(t_i^n - s) + B(t_i^n)\,(s - t_{i-1}^n)}{t_i^n - t_{i-1}^n} \qquad \text{for any } t_{i-1}^n \le s \le t_i^n$$

associated with a convergent partition $0 = t_0^n < t_1^n < \cdots < t_n^n = t$ of $[0, t]$.

Solution

Clearly

$$\begin{aligned}\int_0^t u(B_n(s))\,\dot{B}_n(s)\,ds &= \sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} u(B_n(s))\,\frac{B(t_i^n) - B(t_{i-1}^n)}{t_i^n - t_{i-1}^n}\,ds \\ &= \sum_{i=1}^n [\frac{1}{t_i^n - t_{i-1}^n}\int_{t_{i-1}^n}^{t_i^n} u(B_n(s))\,ds]\,(B(t_i^n) - B(t_{i-1}^n))\end{aligned}$$

fails to converge to the stochastic integral, simply because

$$\frac{1}{t_i^n - t_{i-1}^n}\int_{t_{i-1}^n}^{t_i^n} u(B_n(s))\,ds = \frac{1}{t_i^n - t_{i-1}^n}\int_{t_{i-1}^n}^{t_i^n} u(\frac{B(t_{i-1}^n)\,(t_i^n - s) + B(t_i^n)\,(s - t_{i-1}^n)}{t_i^n - t_{i-1}^n})\,ds$$

is a (complicated) function of the two random variables $B(t_{i-1}^n)$ and $B(t_i^n)$, and cannot be measurable w.r.t. $\mathcal{F}(t_{i-1}^n)$, for any $i = 1, \cdots, n$.

□