INSA Rennes, 4GM–AROM
Random Models of Dynamical Systems
Introduction to SDE's

TD 1 : Brownian motion and continuous martingales

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Exercise 1 Let B be a standard Brownian motion. Show that the processes defined by:

- *rescaling*

$$X(t) = \lambda\, B(\frac{t}{\lambda^2}) \; ,$$

- *time inversion*

$$X(t) = \begin{cases} t\, B(\frac{1}{t}) & \text{if } t > 0 \; , \\ \\ 0 & \text{if } t = 0 \; , \end{cases}$$

- *refreshing*

$$X(t) = B(t + t_0) - B(t_0) \; ,$$

- *time reversal*

$$X(t) = B(T - t) - B(T) \; , \qquad \text{for any } 0 \leq t \leq T \; ,$$

are also standard Brownian motions, i.e. have the same distribution as B.

[Hint for the *time inversion* case: use the law of large numbers for Brownian motion: $\dfrac{B(u)}{u} \to 0$, almost surely as $u \uparrow \infty$.]

Exercise 2 Let B be a standard Brownian motion. Show that B itself, and the processes M and Z defined by

$$M(t) = B^2(t) - t \qquad \text{and} \qquad Z(t) = \exp\{\lambda\, B(t) - \tfrac{1}{2}\, \lambda^2\, t\}$$

are martingales.

Problem 3 [First hitting time for a Brownian motion] Let B be a one–dimensional standard Brownian motion, with $B(0) = 0$. For any $a > 0$, define

$$T_a = \inf\{t \geq 0 \, : \, B(t) \geq a\} \; .$$

(i) **Show that T_a is a stopping time.**

(ii) **For any real number λ and any positive $t > 0$, show that**

$$\mathbb{E}[\,\exp\{\lambda\,B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\}\,] = 1 \ .$$

[Hint: consider the martingale

$$Z^\lambda(t) = \exp\{\lambda\,B(t) - \tfrac{1}{2}\,\lambda^2\,t\} \ ,$$

and use the optional sampling theorem.]

(iii) **Taking $t \uparrow \infty$, show that for any positive $\lambda > 0$**

$$\mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{\lambda\,a - \tfrac{1}{2}\,\lambda^2\,T_a\}\,] = 1 \ .$$

[Hint: consider separately the event $\{T_a < \infty\}$ and its complement $\{T_a = \infty\}$.]

(iv) **Show that $\mathbb{P}[T_a < \infty] = 1$ and show that the Laplace transform of the (probability distribution of the) stopping time T_a is given for any positive $\mu > 0$ by**

$$\mathbb{E}[\,\exp\{-\mu\,T_a\}\,] = \exp\{-\sqrt{2\,\mu}\ a\} \ .$$

Remark: The probability density defined by

$$p_a(t) = \frac{a}{\sqrt{2\pi t^3}}\ \exp\{-\frac{a^2}{2t}\} \ , \qquad \text{for any } t > 0,$$

has Laplace transform $\exp\{-\sqrt{2\,\mu}\ a\}$. In other words, this is the density of the (probability distribution of the) stopping time T_a.

Problem 4 [Brownian bridge] Let B be a one–dimensional standard Brownian motion, with $B(0) = 0$. Introduce the Brownian bridge as the process Z defined by $Z(t) = B(t) - t\,B(1)$, for any $0 \leq t \leq 1$.

(i) **Show that Z is a Gaussian process with zero mean, independent of the random variable $B(1)$.**

(ii) **Give the expression of its correlation function, defined as $K(t,s) = \mathbb{E}[Z(t)\,Z(s)]$ for any $0 \leq s,t \leq 1$.**

(iii) **Show that the process Z' defined by $Z'(t) = Z(1-t)$, for any $0 \leq t \leq 1$, has the same distribution as the Brownian bridge.**

Consider the process Z'' defined by $Z''(t) = (1-t)\,B(\dfrac{t}{1-t})$, for any $0 \leq t < 1$.

(iv) **Show that $Z''(t) \to 0$ almost surely as $t \to 1$ (and define $Z''(1) = 0$ by continuity). Show that Z'' has the same distribution as the Brownian bridge.**

[Hint: use the law of large numbers for Brownian motion: $\dfrac{B(u)}{u} \to 0$, almost surely as $u \uparrow \infty$.]

(v) **Let F be a real–valued bounded continuous mapping defined on the functional space $C([0,1],\mathbb{R})$ of all real–valued continuous functions defined on $[0,1]$. Show that**

$$\mathbb{E}[F(B) \mid |B(1)| < \varepsilon] \to \mathbb{E}[F(Z)] \ ,$$

as $\varepsilon \to 0$.

[Hint: Write B as a continuous function of the pair $(Z, B(1))$.]

Problem 5 [Maximum value of a Brownian bridge] Let B be a one–dimensional standard Brownian motion, with $B(0) = 0$. Recall that the Brownian bridge is the process Z defined by $Z(t) = B(t) - t\,B(1)$, for any $0 \le t \le 1$. Clearly, $Z(0) = Z(1) = 0$, and to assess how far away from zero can the Brownian bridge reach, a natural idea is to introduce the random variable $U = \max\limits_{0 \le t \le 1} Z(t)$ and to let $F(a) = \mathbb{P}[U < a]$.

(i) **Show that $U \ge 0$, and give the expression of $F(a)$ for nonpositive values $a \le 0$.**

From now on, it is assumed that $a > 0$.

(ii) **Show that**

$$1 - F(a) = \mathbb{P}[Z(t) = a, \text{ for some } 0 < t < 1] = \mathbb{P}[B(t) - a\,t = a, \text{ for some } t > 0] \ .$$

[Hint: introduce the process defined by $Z''(t) = (1-t)\,B(\dfrac{t}{1-t})$, for any $0 \le t < 1$.]

For any $a > 0$, define

$$T_a = \inf\{t \ge 0 \ : \ B(t) - a\,t \ge a\} \ .$$

(iii) **Show that T_a is a stopping time and that**

$$1 - F(a) = \mathbb{P}[T_a < \infty] \ .$$

(iv) **For any positive $t > 0$, show that**

$$\mathbb{E}[\exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t))\}] = 1 \ .$$

[Hint: consider the martingale

$$Z^\lambda(t) = \exp\{\lambda\,B(t) - \tfrac{1}{2}\,\lambda^2\,t\} \ ,$$

for $\lambda = 2\,a$, and use the optional sampling theorem.]

(v) **Taking $t \uparrow \infty$, show that**

$$\mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{2\,a\,B(T_a) - 2\,a^2\,T_a\}\,] = 1\ .$$

[Hint: consider separately the event $\{T_a < \infty\}$ and its complement $\{T_a = \infty\}$.]

(vi) **Conclude that**

$$\mathbb{P}[T_a < \infty] = \exp\{-2\,a^2\}\ ,$$

and give the expression of (i) the cumulative distribution function and (ii) the probability density function of the random variable U.