INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems
Introduction to SDE's

TD 1 : Brownian motion and continuous martingales

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Exercise 0 [Law of large numbers]

Let B be a standard Brownian motion. Then

$$\frac{B(t)}{t} \to 0 \; ,$$

almost surely as $t \uparrow \infty$.

Solution

For any $t' \leq t \leq t''$, it holds

$$\frac{|B(t)|}{t} \leq \frac{1}{t'} \, |B(t)| \; ,$$

hence

$$\max_{t' \leq t \leq t''} \frac{|B(t)|}{t} \leq \frac{1}{t'} \max_{t' \leq t \leq t''} |B(t)| \leq \frac{1}{t'} \max_{0 \leq t \leq t''} |B(t)| \; ,$$

and the Doob maximal inequality yields

$$\mathbb{P}[\max_{t' \leq t \leq t''} \frac{|B(t)|}{t} \geq \varepsilon] \leq \mathbb{P}[\max_{0 \leq t \leq t''} |B(t)| \geq \varepsilon \, t'] \leq \frac{1}{(\varepsilon \, t')^2} \, \mathbb{E}|B(t'')|^2 \leq \frac{t''}{(\varepsilon \, t')^2} \; .$$

Taking $t' = 2^n$ and $t'' = 2^{n+1}$, it holds

$$\mathbb{P}[\max_{2^n \leq t \leq 2^{n+1}} \frac{|B(t)|}{t} \geq \varepsilon] \leq \frac{2^{n+1}}{(\varepsilon \, 2^n)^2} = \frac{1}{\varepsilon^2} \, 2^{-n+1} \; ,$$

and the Borel–Cantelli lemma yields

$$\mathbb{P}[\bigcap_{p \geq 0} \bigcup_{n \geq p} \{ \max_{2^n \leq t \leq 2^{n+1}} \frac{|B(t)|}{t} \geq \varepsilon \}] = 0 \; ,$$

or

$$\begin{aligned} \mathbb{P}[\bigcup_{p \geq 0} \bigcap_{n \geq p} \{ \max_{2^n \leq t \leq 2^{n+1}} \frac{|B(t)|}{t} < \varepsilon \}] &= \mathbb{P}[\bigcup_{p \geq 0} \{ \max_{t \geq 2^p} \frac{|B(t)|}{t} < \varepsilon \}] \\ &= \mathbb{P}[\max_{t \geq 2^p} \frac{|B(t)|}{t} < \varepsilon \text{ for some } p \geq 0] \\ &= \mathbb{P}[\max_{t \geq t_0} \frac{|B(t)|}{t} < \varepsilon \text{ for some } t_0 \geq 0] \\ &= \mathbb{P}[\limsup_{t \uparrow \infty} \frac{|B(t)|}{t} < \varepsilon] = 1 \; , \end{aligned}$$

and its is easy to conclude that $\frac{B(t)}{t} \to 0$ almost surely as $t \uparrow \infty$.

A simple change of variable shows that $t\,B(\frac{1}{t}) \to 0$ almost surely as $t \downarrow 0$. $\square$

Exercise 1 Let B be a standard Brownian motion. Show that the processes defined by:

- *rescaling*

$$X(t) = \lambda\, B(\frac{t}{\lambda^2})\ ,$$

- *time inversion*

$$X(t) = \begin{cases} t\,B(\frac{1}{t}) & \text{if } t > 0\ , \\ 0 & \text{if } t = 0\ , \end{cases}$$

- *refreshing*

$$X(t) = B(t + t_0) - B(t_0)\ ,$$

- *time reversal*

$$X(t) = B(T - t) - B(T)\ , \qquad \text{for any } 0 \le t \le T\ ,$$

are also standard Brownian motions, i.e. have the same distribution as B.

[Hint for the *time inversion* case: use the law of large numbers for Brownian motion: $\frac{B(u)}{u} \to 0$, almost surely as $u \uparrow \infty$.]

Solution

It is convenient here to use the following criterion: a process B is a standard Brownian motion iff B is a zero mean Gaussian process with correlation function

$$K(s,t) = \mathbb{E}[B(t)\,B(s)] = s \wedge t\ ,$$

and almost surely continuous sample paths.

For the *rescaling* case: Clearly, the process X is Gaussian and has almost surely continuous sample paths. Moreover

$$\begin{aligned} \mathbb{E}[X(t)\,X(s)] &= \lambda^2\,\mathbb{E}[B(\frac{t}{\lambda^2})\,B(\frac{s}{\lambda^2})] \\ &= \lambda^2\,\min(\frac{t}{\lambda^2}, \frac{s}{\lambda^2}) \\ &= \min(t,s)\ . \end{aligned}$$

For the *time inversion* case: Note that the mapping $u \to \dfrac{1}{u}$ is decreasing, hence

$$\min(\frac{1}{t}, \frac{1}{s}) = \frac{1}{\max(t,s)} ,$$

and

$$\begin{aligned}
\mathbb{E}[X(t)\,X(s)] &= t\,s\;\mathbb{E}[B(\frac{1}{t})\,B(\frac{1}{s})] \\
&= t\,s\;\min(\frac{1}{t}, \frac{1}{s}) \\
&= \max(t,s)\;\min(t,s)\;\frac{1}{\max(t,s)} \\
&= \min(t,s)\;.
\end{aligned}$$

For the *refreshing* case: Clearly, the process X is Gaussian and has almost surely continuous sample path. Assuming that $0 \le s \le t$ without loss of generality, it holds $t_0 \le t_0 + s \le t_0 + t$ hence $(B(t_0+t) - B(t_0+s))$ and $(B(t_0+s) - B(t_0))$ are independent r.v.'s and

$$\begin{aligned}
\mathbb{E}[X(t)\,X(s)] &= \mathbb{E}[(B(t_0+t) - B(t_0))\,(B(t_0+s) - B(t_0))] \\
&= \mathbb{E}[(B(t_0+t) - B(t_0+s))\,(B(t_0+s) - B(t_0))] + \mathbb{E}[(B(t_0+s) - B(t_0))^2] \\
&= (t_0+s) - t_0 = s\;.
\end{aligned}$$

Alternatively, simple expansion yields

$$\begin{aligned}
\mathbb{E}[X(t)\,X(s)] &= \mathbb{E}[(B(t_0+t) - B(t_0))\,(B(t_0+s) - B(t_0))] \\
&= \mathbb{E}[B(t_0+t)\,B(t_0+s)] - \mathbb{E}[B(t_0+t)\,B(t_0)] - \mathbb{E}[B(t_0)\,B(t_0+s)] + \mathbb{E}[B^2(t_0)] \\
&= (t_0+s) - t_0 - t_0 + t_0 = s\;.
\end{aligned}$$

For the *time reversal* case: Clearly, the process X is Gaussian and has almost surely continuous sample path. Assuming that $0 \le s \le t \le T$ without loss of generality, it holds $0 \le T - t \le T - s \le T$ hence $(B(T) - B(T-s))$ and $(B(T-s) - B(T-t))$ are independent r.v.'s and

$$\begin{aligned}
\mathbb{E}[X(t)\,X(s)] &= \mathbb{E}[(B(T-t) - B(T))\,(B(T-s) - B(T))] \\
&= \mathbb{E}[(B(T) - B(T-t))\,(B(T) - B(T-s))] \\
&= \mathbb{E}[(B(T) - B(T-s))^2] + \mathbb{E}[(B(T-s) - B(T-t))\,(B(T) - B(T-s))] \\
&= T - (T-s) = s\;.
\end{aligned}$$

Alternatively, simple expansion yields

$$\begin{aligned}\mathbb{E}[X(t)\,X(s)] &= \mathbb{E}[(B(T-t)-B(T))\,(B(T-s)-B(T))] \\ &= \mathbb{E}[B(T-t)\,B(T-s)] - \mathbb{E}[B(T-t)\,B(T)] - \mathbb{E}[B(T)\,B(T-s)] + \mathbb{E}[B^2(T)] \\ &= (T-t)-(T-t)-(T-s)+T = s\ .\end{aligned}$$

□

Exercise 2 Let B be a standard Brownian motion. Show that B itself, and the processes M and Z defined by

$$M(t) = B^2(t) - t \qquad \text{and} \qquad Z(t) = \exp\{\lambda\, B(t) - \tfrac{1}{2}\,\lambda^2\, t\}$$

are martingales.

SOLUTION

For any $0 \le s \le t$, the r.v. $(B(t)-B(s))$ is zero mean and is independent of $\mathcal{F}(s)$, hence

$$\mathbb{E}[B(t) \mid \mathcal{F}(s)] - B(s) = \mathbb{E}[B(t)-B(s) \mid \mathcal{F}(s)] = 0\ ,$$

i.e. B is a martingale.

For any $0 \le s \le t$

$$M(t)-M(s) = (B^2(t)-B^2(s)) - (t-s) = (B(t)-B(s))^2 - (t-s) + 2\,B(s)\,(B(t)-B(s))\ ,$$

and the r.v. $(B(t)-B(s))$ is zero mean with variance $(t-s)$ and is independent of $\mathcal{F}(s)$, hence

$$\begin{aligned}\mathbb{E}[M(t) \mid \mathcal{F}(s)] - M(s) &= \mathbb{E}[M(t)-M(s) \mid \mathcal{F}(s)] \\ &= \mathbb{E}[(B(t)-B(s))^2 \mid \mathcal{F}(s)] - (t-s) + 2\,B(s)\,\mathbb{E}[B(t)-B(s) \mid \mathcal{F}(s)] = 0\ ,\end{aligned}$$

i.e. M is a martingale.

For any $0 \le s \le t$

$$Z(t) = \exp\{\lambda\,(B(t)-B(s))\}\ \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\}\ Z(s)\ ,$$

and the r.v. $(B(t)-B(s))$ is Gaussian, with zero mean and variance $(t-s)$ and is independent of $\mathcal{F}(s)$, hence the Laplace transform

$$\mathbb{E}[\exp\{\lambda\,(B(t)-B(s))\} \mid \mathcal{F}(s)] = \exp\{\tfrac{1}{2}\,\lambda^2\,(t-s)\}\ ,$$

and

$$\mathbb{E}[Z(t) \mid \mathcal{F}(s)] = \mathbb{E}[\exp\{\lambda\,(B(t)-B(s))\} \mid \mathcal{F}(s)]\ \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\}\ Z(s) = Z(s)\ ,$$

i.e. Z is a martingale.

□

Problem 3 [First hitting time for a Brownian motion] Let B be a one–dimensional standard Brownian motion, with $B(0) = 0$. For any $a > 0$, define

$$T_a = \inf\{t \geq 0 \,:\, B(t) \geq a\} \ .$$

(i) **Show that T_a is a stopping time.**

Solution

By definition, the event $\{T_a \leq t\} = \{B(s) \geq a \text{ for some } 0 \leq s \leq t\}$ is measurable w.r.t. the σ–algebra $\mathcal{F}(t) = \sigma(B(s)\,,\, 0 \leq s \leq t)$, i.e. the random variable T_a is a stopping time.

□

(ii) **For any real number λ and any positive $t > 0$, show that**

$$\mathbb{E}[\,\exp\{\lambda\, B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\}\,] = 1 \ .$$

[Hint: consider the martingale

$$Z^\lambda(t) = \exp\{\lambda\, B(t) - \tfrac{1}{2}\,\lambda^2\, t\} \ ,$$

and use the optional sampling theorem.]

Solution

Introducing the martingale

$$Z^\lambda(t) = \exp\{\lambda\, B(t) - \tfrac{1}{2}\,\lambda^2\, t\} \ ,$$

and using the optional sampling theorem with the bounded stopping time $T_a \wedge t$, yields

$$\mathbb{E}[\,\exp\{\lambda\, B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\}\,] = 1 \ .$$

□

(iii) **Taking $t \uparrow \infty$, show that for any positive $\lambda > 0$**

$$\mathbb{E}[\,1_{(T_a\,<\,\infty)}\, \exp\{\lambda\, a - \tfrac{1}{2}\,\lambda^2\, T_a\}\,] = 1 \ .$$

[Hint: consider separately the event $\{T_a < \infty\}$ and its complement $\{T_a = \infty\}$.]

Solution

Clearly

$$1_{(T_a\,<\,\infty)}\, \exp\{\lambda\, B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\} \to 1_{(T_a\,<\,\infty)}\, \exp\{\lambda\, B(T_a) - \tfrac{1}{2}\,\lambda^2\, T_a\} \ ,$$

almost surely as $t \uparrow \infty$, and

$$1_{(T_a\,=\,\infty)}\, \exp\{\lambda\, B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\} = 1_{(T_a\,=\,\infty)}\, \exp\{\lambda\, B(t) - \tfrac{1}{2}\,\lambda^2\, t\}$$

$$= 1_{(T_a\,=\,\infty)}\, \exp\{\lambda\, t\,(\frac{B(t)}{t} - \tfrac{1}{2}\,\lambda)\} \to 0 \ ,$$

almost surely as $t \uparrow \infty$. Note that for any $0 \le s \le T_a$ (and in particular for $s = T_a \wedge t$) it holds $B(s) \le a$, hence for any positive $\lambda > 0$

$$\exp\{\lambda\, B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\} \le \exp\{\lambda\, a\}\ ,$$

and convergence holds also in L^1, using the Lebesgue dominated convergence theorem. Therefore

$$\begin{aligned} 1 \ &= \ \mathbb{E}[\,\exp\{\lambda\, B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\}\,] \\ &= \ \mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{\lambda\, B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\}\,] \\ &\qquad + \mathbb{E}[\,1_{(T_a\,=\,\infty)}\ \exp\{\lambda\, B(T_a \wedge t) - \tfrac{1}{2}\,\lambda^2\,(T_a \wedge t)\}\,] \\ &\to \ \mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{\lambda\, B(T_a) - \tfrac{1}{2}\,\lambda^2\,T_a\}\,]\ . \end{aligned}$$

Clearly $B(T_a) = a$, hence

$$1 = \mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{\lambda\, B(T_a) - \tfrac{1}{2}\,\lambda^2\,T_a\}\,] = \mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{\lambda\, a - \tfrac{1}{2}\,\lambda^2\,T_a\}\,]\ ,$$

or equivalently

$$\mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{-\tfrac{1}{2}\,\lambda^2\,T_a\}\,] = \exp\{-\lambda\, a\}\ .$$

□

(iv) **Show that $\mathbb{P}[T_a < \infty] = 1$ and show that the Laplace transform of the (probability distribution of the) stopping time T_a is given for any positive $\mu > 0$ by**

$$\mathbb{E}[\,\exp\{-\mu\, T_a\}\,] = \exp\{-\sqrt{2\,\mu}\ a\}\ .$$

Solution

Clearly

$$1_{(T_a\,<\,\infty)}\ \exp\{-\tfrac{1}{2}\,\lambda^2\,T_a\} \to 1_{(T_a\,<\,\infty)}\ ,$$

almost surely as $\lambda \downarrow 0$, and note that

$$1_{(T_a\,<\,\infty)}\ \exp\{-\tfrac{1}{2}\,\lambda^2\,T_a\} \le 1\ ,$$

and convergence holds also in L^1, using the Lebesgue dominated convergence theorem. Therefore

$$\mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{-\tfrac{1}{2}\,\lambda^2\,T_a\}\,] \to \mathbb{P}[T_a < \infty]\ ,$$

and

$$\mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{-\tfrac{1}{2}\,\lambda^2\,T_a\}\,] = \exp\{-\lambda\, a\} \to 1\ ,$$

as $\lambda \downarrow 0$, and uniqueness of the limit yields $\mathbb{P}[T_a < \infty] = 1$.

Therefore

$$\mathbb{E}[\,\exp\{-\tfrac{1}{2}\,\lambda^2\,T_a\}\,] = \mathbb{E}[\,1_{(T_a\,<\,\infty)}\ \exp\{-\tfrac{1}{2}\,\lambda^2\,T_a\}\,] = \exp\{-\lambda\, a\}\ ,$$

and taking $\lambda = \sqrt{2\,\mu}$ yields

$$\mathbb{E}[\,\exp\{-\mu\, T_a\}\,] = \exp\{-\sqrt{2\,\mu}\ a\}\ .$$

□

Remark: The probability density defined by

$$p_a(t) = \frac{a}{\sqrt{2\pi t^3}}\ \exp\{-\frac{a^2}{2t}\}\ , \qquad \text{for any } t > 0,$$

has Laplace transform $\exp\{-\sqrt{2\,\mu}\ a\}$. In other words, this is the density of the (probability distribution of the) stopping time T_a.

(v) **Show that the stopping time T_a has the same distribution as the r.v. $\dfrac{a^2}{X^2}$ where X is a standard Gaussian r.v.**

SOLUTION

Using the change of variable $t = \dfrac{a^2}{x^2}$, with $dt = 2\,\dfrac{a^2}{x^3}\,dx$, it holds

$$\sqrt{t^3} = \frac{a^3}{x^3} \qquad \text{and} \qquad \frac{a^2}{t} = x^2\ ,$$

hence

$$\begin{aligned}
\mathbb{E}[\phi(T_a)] &= \int_0^\infty \phi(t)\ p_a(t)\,dt \\
&= \int_0^\infty \phi(t)\ \frac{a}{\sqrt{2\pi t^3}}\ \exp\{-\frac{a^2}{2t}\}\,dt \\
&= \int_0^\infty \phi(\frac{a^2}{x^2})\ \frac{a\,x^3}{\sqrt{2\pi}\,a^3}\ \exp\{-\tfrac{1}{2}\,x^2\}\,2\,\frac{a^2}{x^3}\,dx \\
&= 2\,\frac{1}{\sqrt{2\pi}}\int_0^\infty \phi(\frac{a^2}{x^2})\ \exp\{-\tfrac{1}{2}\,x^2\}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \phi(\frac{a^2}{x^2})\ \exp\{-\tfrac{1}{2}\,x^2\}\,dx \\
&= \mathbb{E}[\phi(\frac{a^2}{X^2})]\ ,
\end{aligned}$$

for any bounded measurable function ϕ.

□

Problem 4 [Brownian bridge] Let B be a one–dimensional standard Brownian motion, with $B(0) = 0$. Introduce the Brownian bridge as the process Z defined by $Z(t) = B(t) - t\,B(1)$, for any $0 \leq t \leq 1$.

(i) **Show that Z is a Gaussian process with zero mean, independent of the random variable $B(1)$.**

Solution

Clearly

$$\mathbb{E}[Z(t)] = \mathbb{E}[B(t)] - t\,\mathbb{E}[B(1)] = 0\ ,$$

for any $0 \le t \le 1$.

For any integer $n \ge 1$ and any time instants $0 \le t_1 < \cdots < t_n \le 1$, the vector $(Z(t_1), \cdots, Z(t_n))$ is a linear transformation of the Gaussian random vector $(B(t_1), \cdots, B(t_n), B(1))$, hence it is a Gaussian random vector. This shows that the whole process Z is Gaussian.

Clearly

$$\mathbb{E}[Z(t)\,B(1)] = \mathbb{E}[(B(t) - t\,B(1))\,B(1)] = \mathbb{E}[B(t)\,B(1)] - t\,\mathbb{E}[B^2(1)] = 0\ ,$$

hence the two Gaussian random variables $Z(t)$ and $B(1)$ are independent, since they have zero correlation.

□

(ii) **Give the expression of its correlation function, defined as $K(t,s) = \mathbb{E}[Z(t)\,Z(s)]$ for any $0 \le s,t \le 1$.**

Solution

By definition

$$\begin{aligned}
K(t,s) &= \mathbb{E}[Z(t)\,Z(s)] \\
&= \mathbb{E}[(B(t) - t\,B(1))\,(B(s) - s\,B(1))] \\
&= \mathbb{E}[B(t)\,B(s)] - s\,E[B(t)\,B(1)] - t\,\mathbb{E}[B(s)\,B(1)] + t\,s\,\mathbb{E}[B^2(1)] \\
&= \min(t,s) - s\,t \\
&= \min(t,s) - \min(t,s)\ \max(t,s) \\
&= \min(t,s)\ (1 - \max(t,s))\ .
\end{aligned}$$

□

(iii) **Show that the process Z' defined by $Z'(t) = Z(1-t)$, for any $0 \le t \le 1$, has the same distribution as the Brownian bridge.**

Solution

Note that the mapping $u \mapsto 1-u$ is decreasing, hence

$$\min(1-t, 1-s) = 1 - \max(t,s) \qquad \text{and} \qquad \max(1-t, 1-s) = 1 - \min(t,s)\ .$$

By definition

$$\begin{aligned} K'(t,s) &= \mathbb{E}[Z'(t)\, Z'(s)] \\ &= \mathbb{E}[Z(1-t)\, Z(1-s)] \\ &= \min(1-t, 1-s)\; (1 - \max(1-t, 1-s)) \\ &= (1 - \max(t,s))\; (1 - (1 - \min(t,s))) \\ &= \min(t,s)\; (1 - \max(t,s)) \; . \end{aligned}$$

Clearly, the process Z' is Gaussian, has almost surely continuous sample paths, and its correlation function coincides with the correlation function of the Brownian bridge Z. Therefore, the two processes Z and Z' have the same finite–dimensional distributions, hence they have the same distribution.

□

Consider the process Z'' defined by $Z''(t) = (1-t)\, B(\frac{t}{1-t})$, for any $0 \leq t < 1$.

(iv) **Show that $Z''(t) \to 0$ almost surely as $t \to 1$ (and define $Z''(1) = 0$ by continuity). Show that Z'' has the same distribution as the Brownian bridge.**

[Hint: use the law of large numbers for Brownian motion: $\dfrac{B(u)}{u} \to 0$, almost surely as $u \uparrow \infty$.]

Solution

Clearly

$$Z''(t) = (1-t)\, B(\frac{t}{1-t}) = t\; \frac{B(\dfrac{t}{1-t})}{\dfrac{t}{1-t}} \; ,$$

and using the time change $u = \dfrac{t}{1-t}$, shows that

$$\lim_{t \to 1} \frac{B(\dfrac{t}{1-t})}{\dfrac{t}{1-t}} = \lim_{u \to \infty} \frac{B(u)}{u} = 0 \; ,$$

almost surely, hence $Z''(t) \to 0$ almost surely as $t \to 1$.

Note that the mapping $u \mapsto \dfrac{u}{1-u}$ is increasing, hence

$$\mathbb{E}[B(\frac{t}{1-t})\, B(\frac{s}{1-s})] = \min(\frac{t}{1-t}, \frac{s}{1-s}) = \frac{\min(t,s)}{1 - \min(t,s)} \; ,$$

and

$$\begin{aligned} K''(t,s) &= \mathbb{E}[Z''(t)\, Z''(s)] \\ &= (1-t)\,(1-s)\; \mathbb{E}[B(\frac{t}{1-t})\, B(\frac{s}{1-s})] \\ &= (1-\min(t,s))\,(1-\max(t,s))\; \frac{\min(t,s)}{1-\min(t,s)} \\ &= \min(t,s)\,(1-\max(t,s)) \ . \end{aligned}$$

Clearly, the process Z'' is Gaussian, has almost surely continuous sample paths, and its correlation function coincides with the correlation function of the Brownian bridge Z. Therefore, the two processes Z and Z'' have the same finite–dimensional distributions, hence they have the same distribution.

□

(v) **Let F be a real–valued bounded continuous mapping defined on the functional space $C([0,1],\mathbb{R})$ of all real–valued continuous functions defined on $[0,1]$. Show that**

$$\mathbb{E}[F(B) \mid |B(1)| < \varepsilon] \to \mathbb{E}[F(Z)] \ ,$$

as $\varepsilon \to 0$.

[Hint: Write B as a continuous function of the pair $(Z, B(1))$.]

Solution

Let Φ denote the mapping defined on $C([0,1],\mathbb{R}) \times \mathbb{R}$ and taking values in $C([0,1],\mathbb{R})$, such that for any $u \in C([0,1],\mathbb{R})$ and any $\alpha \in \mathbb{R}$, the resulting $\Phi(u,\alpha) \in C([0,1],\mathbb{R})$ is defined by

$$\Phi(u,\alpha)(t) = u(t) + t\,\alpha \ , \qquad \text{for any } 0 \le t \le 1.$$

Clearly Φ is a continuous mapping, and the definition $Z(t) = B(t) - t\,B(1)$ for any $0 \le t \le 1$ implies $B = \Phi(Z, B(1))$. Therefore

$$\begin{aligned} \mathbb{E}[F(B) \mid |B(1)| < \varepsilon] &= \mathbb{E}[F(\Phi(Z,B(1))) \mid |B(1)| < \varepsilon] \\ &= \frac{\mathbb{E}[F(\Phi(Z,B(1)))\, 1_{(|B(1)|<\varepsilon)}]}{\mathbb{P}[|B(1)| < \varepsilon]} \end{aligned}$$

Recall that Z and $B(1)$ are independent, and $B(1)$ is a standard Gaussian random variable (with mean zero and variance unity), hence

$$\begin{aligned} \mathbb{E}[F(\Phi(Z,B(1)))\, 1_{(|B(1)|<\varepsilon)}] &= \int \mathbb{E}[F(\Phi(Z,x))]\, 1_{(|x|<\varepsilon)}\, \frac{1}{\sqrt{2\pi}}\, \exp\{-\tfrac{1}{2}\, x^2\}\, dx \\ &= \frac{\varepsilon}{\sqrt{2\pi}} \int_{-1}^{1} \mathbb{E}[F(\Phi(Z,\varepsilon\, y))]\, \exp\{-\tfrac{1}{2}\,\varepsilon^2\, y^2\}\, dy \ . \end{aligned}$$

Clearly

$$F(\Phi(Z,\varepsilon\, y))\ \exp\{-\tfrac{1}{2}\,\varepsilon^2\, y^2\} \to F(\Phi(Z,0)) = F(Z)\ ,$$

almost everywhere as $\varepsilon \downarrow 0$, hence

$$\tfrac{1}{2}\int_{-1}^{1} \mathbb{E}[F(\Phi(Z,\varepsilon\, y))]\ \exp\{-\tfrac{1}{2}\,\varepsilon^2\, y^2\}\, dy \to \mathbb{E}[F(Z)]\ ,$$

as $\varepsilon \downarrow 0$, using the Lebesgue dominated convergence theorem, and also

$$\tfrac{1}{2}\int_{-1}^{1} \exp\{-\tfrac{1}{2}\,\varepsilon^2\, y^2\}\, dy \to 1\ ,$$

hence

$$\mathbb{E}[F(B) \mid |B(1)| < \varepsilon] \to \mathbb{E}[F(Z)]\ ,$$

as $\varepsilon \downarrow 0$.

Complement: More generally

$$\begin{aligned}\mathbb{E}[F(B)\ 1_{(B(1)\,\in\, A)}] &= \mathbb{E}[F(\Phi(Z,B(1)))\ 1_{(B(1)\,\in\, A)}] \\ &= \int_A \mathbb{E}[F(\Phi(Z,x))]\ \frac{1}{\sqrt{2\pi}}\ \exp\{-\tfrac{1}{2}\,x^2\}\, dx\ ,\end{aligned}$$

for any Borel subset A, hence

$$\mathbb{E}[F(\Phi(Z,x))] = \mathbb{E}[F(B) \mid B(1) = x]\ ,$$

for almost every x. Note that

$$F(\phi(Z,x')) \to F(\phi(Z,x))\ ,$$

almost surely as $x' \to x$, and convergence holds also in L^1, using the Lebesgue dominated convergence theorem, i.e.

$$\mathbb{E}[F(\Phi(Z,x'))] \to \mathbb{E}[F(\phi(Z,x))]\ ,$$

as $x' \to x$. In other words, the mapping $x \mapsto \mathbb{E}[F(\Phi(Z,x))]$ is continuous, and $\mathbb{E}[F(\Phi(Z,x))]$ can be seen a continuous version of the conditional expectation $\mathbb{E}[F(B) \mid B(1) = x]$. In particular for $x = 0$, it holds

$$\mathbb{E}[F(Z)] = \mathbb{E}[F(\Phi(Z,0))] = \mathbb{E}[F(B) \mid B(1) = 0]\ .$$

i.e. the distribution of the Brownian bridge Z is the same as the conditional distribution of the Brownian motion conditioned by taking the value 0 at time 1.

□

Problem 5 [Maximum value of a Brownian bridge] Let B be a one–dimensional standard Brownian motion, with $B(0) = 0$. Recall that the Brownian bridge is the process Z defined by $Z(t) = B(t) - t\,B(1)$, for any $0 \leq t \leq 1$. Clearly, $Z(0) = Z(1) = 0$, and to assess how far away from zero can the Brownian bridge reach, a natural idea is to introduce the random variable $U = \max\limits_{0 \leq t \leq 1} Z(t)$ and to let $F(a) = \mathbb{P}[U < a]$.

(i) **Show that $U \geq 0$, and give the expression of $F(a)$ for nonpositive values $a \leq 0$.**

Solution

The maximum $\max\limits_{0 \leq t \leq 1} Z(t)$ is larger than any particular value such as $Z(0)$, hence $U \geq 0$.

For any nonpositive $a \leq 0$, it holds $\{U < a\} \subseteq \{U < 0\} = \emptyset$, hence $F(a) = \mathbb{P}[U < a] = 0$.

□

From now on, it is assumed that $a > 0$.

(ii) **Show that**

$$1 - F(a) = \mathbb{P}[Z(t) = a, \text{ for some } 0 < t < 1] = \mathbb{P}[B(t) - a\,t = a, \text{ for some } t > 0] \ .$$

[Hint: introduce the process defined by $Z''(t) = (1-t)\,B(\frac{t}{1-t})$, for any $0 \leq t < 1$.]

Solution

Clearly, the level a cannot be reached for $t = 0$ nor for $t = 1$, since $Z(0) = Z(1) = 0$, hence

$$U = \max_{0 \leq t \leq 1} Z(t) = \max_{0 < t < 1} Z(t) \ .$$

By definition, and using the change of variable $s = \dfrac{t}{1-t}$ (so that $t = \dfrac{s}{1+s}$ and $1 - t = \dfrac{1}{1+s}$), it holds

$$\begin{aligned}
1 - F(a) &= \mathbb{P}[\max_{0<t<1} Z(t) \geq a] \\
&= \mathbb{P}[Z(t) = a, \text{ for some } 0 < t < 1] \\
&= \mathbb{P}[(1-t)\,B(\frac{t}{1-t}) = a, \text{ for some } 0 < t < 1] \\
&= \mathbb{P}[B(s) = a\,(1+s), \text{ for some } s > 0] \\
&= \mathbb{P}[B(s) - a\,s = a, \text{ for some } s > 0] \ .
\end{aligned}$$

□

For any $a > 0$, define

$$T_a = \inf\{t \geq 0 \,:\, B(t) - a\,t \geq a\} \ .$$

(iii) **Show that T_a is a stopping time and that**

$$1 - F(a) = \mathbb{P}[T_a < \infty] \ .$$

Solution

By definition, the event $\{T_a \le t\} = \{B(s) - a\,s \ge a, \ \text{for some } 0 \le s \le t\}$ is measurable w.r.t. the σ–algebra $\mathcal{F}(t) = \sigma(B(s) \, : \, 0 \le s \le t)$, i.e. the random variable T_a is a stopping time.

Using the answer to the previous question yields

$$\begin{aligned} \mathbb{P}[T_a < \infty] \ &= \ \mathbb{P}[B(t) - a\,t \ge a, \ \text{for some } t > 0] \\ &= \ \mathbb{P}[B(t) - a\,t = a, \ \text{for some } t > 0] = 1 - F(a) \ . \end{aligned}$$

□

(iv) **For any positive $t > 0$, show that**

$$\mathbb{E}[\exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t))\}] = 1 \ .$$

[Hint: consider the martingale

$$Z^\lambda(t) = \exp\{\lambda\,B(t) - \tfrac{1}{2}\,\lambda^2\,t\} \ ,$$

for $\lambda = 2\,a$, and use the optional sampling theorem.]

Solution

Introducing the martingale

$$Z^\lambda(t) = \exp\{\lambda\,B(t) - \tfrac{1}{2}\,\lambda^2\,t\} \ ,$$

for $\lambda = 2\,a$, and using the optional sampling theorem with the bounded stopping time $T_a \wedge t$, yields

$$\mathbb{E}[\exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t)\}] = 1 \ .$$

□

(v) **Taking $t \uparrow \infty$, show that**

$$\mathbb{E}[1_{(T_a < \infty)} \ \exp\{2\,a\,B(T_a) - 2\,a^2\,T_a\}] = 1 \ .$$

[Hint: consider separately the event $\{T_a < \infty\}$ and its complement $\{T_a = \infty\}$.]

Solution

Clearly

$$1_{(T_a < \infty)} \ \exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t)\} \to 1_{(T_a < \infty)} \ \exp\{2\,a\,B(T_a) - 2\,a^2\,T_a\} \ ,$$

almost surely as $t \uparrow \infty$, and

$$1_{(T_a = \infty)} \exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t)\} = 1_{(T_a = \infty)} \exp\{2\,a\,B(t) - 2\,a^2\,t\}$$

$$= 1_{(T_a = \infty)} \exp\{2\,a\,t\,(\frac{B(t)}{t} - a)\} \to 0 \ ,$$

almost surely as $t \uparrow \infty$. Note that for any $0 \leq s \leq T_a$ (and in particular for $s = T_a \wedge t$) it holds $B(s) - a\,s \leq a$, hence

$$\exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t)\} = \exp\{2\,a\,(B(T_a \wedge t) - a\,(T_a \wedge t))\} \leq \exp\{2\,a^2\} \ ,$$

and convergence holds also in L^1, using the Lebesgue dominated convergence theorem. Therefore

$$\begin{aligned} 1 \ &= \ \mathbb{E}[\,\exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t)\}\,] \\ &= \ \mathbb{E}[\,1_{(T_a < \infty)} \exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t)\}\,] \\ &\qquad + \mathbb{E}[\,1_{(T_a = \infty)} \exp\{2\,a\,B(T_a \wedge t) - 2\,a^2\,(T_a \wedge t)\}\,] \\ &\to \ \mathbb{E}[\,1_{(T_a < \infty)} \exp\{2\,a\,B(T_a) - 2\,a^2\,T_a\}\,] \ . \end{aligned}$$

□

(vi) **Conclude that**

$$\mathbb{P}[T_a < \infty] = \exp\{-2\,a^2\} \ ,$$

and give the expression of (i) the cumulative distribution function and (ii) the probability density function of the random variable U.

Solution

Clearly $B(T_a) - a\,T_a = a$, hence

$$\begin{aligned} 1 \ &= \ \mathbb{E}[\,1_{(T_a < \infty)} \exp\{2\,a\,B(T_a) - 2\,a^2\,T_a\}\,] \\ &= \ \mathbb{E}[\,1_{(T_a < \infty)} \exp\{2a\,(B(T_a) - a\,T_a)\}\,] \\ &= \ \exp\{2a^2\}\,\mathbb{P}[T_a < \infty] \ , \end{aligned}$$

or in other words

$$1 - F(a) = \mathbb{P}[T_a < \infty] = \exp\{-2a^2\} \ .$$

Therefore

$$\mathbb{P}[U < a] = F(a) = 1 - \exp\{-2a^2\} \ ,$$

and by continuity, the cumulative distribution function is

$$\mathbb{P}[U \leq a] = 1 - \exp\{-2a^2\} \ ,$$

and the probability density function is just the derivative, i.e.

$$p(a) = 4\,a\ \exp\{-2a^2\} \ .$$

□