INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems II
Introduction to SDE's

Written Exam (aka DS)

January 11, 2017

The objective is to study the one–dimensional (Lévy stochastic area) process

$$A(t) = \int_0^t B_1(s)\, dB_2(s) - \int_0^t B_2(s)\, dB_1(s) \ ,$$

where $B(t) = (B_1(t), B_2(t))$ is a two–dimensional standard Brownian motion, with $B(0) = 0$. The Lévy stochastic area is used for instance in the design of high–order numerical schemes for SDE's.

(i) **Show that $A(t)$ is an Itô process (and give its decomposition in terms of a usual integral and a stochastic integral).**

(ii) **Show that $\mathbb{E}[A(t)] = 0$ and $\mathbb{E}[|A(t)|^2] = t^2$.**

To go further, i.e. beyond the expression of the first two moments, and to study the probability distribution of the r.v. $A(t)$, it is convenient to introduce the one–dimensional processes

$$C(t) = B_1^2(t) + B_2^2(t) = |B(t)|^2 \qquad \text{and} \qquad D(t) = B_1(t)\, B_2(t) \ .$$

In particular, it will be proved that the charactristic function satisfies

$$\mathbb{E}[\exp\{i\, u\, A(t)\}] = \frac{1}{\cosh(u\, t)} \ , \qquad (\star)$$

for any real number u. Here, 'cosh' denotes the hyperbolic cosine function.

(iii) **Show that $C(t)$ and $D(t)$ are two Itô processes (and give their decompositions in terms of a usual integral and a stochastic integral).**

(iv) **Check that $A(t)$ and $-A(t)$ have the same probability distribution, hence the probability distribution is symmetric. Show that**

$$\mathbb{E}[\exp\{i\, u\, A(t)\}] = \mathbb{E}[\cos(u\, A(t))] \ ,$$

for any real number u.

Define also the one–dimensional processes

$$\begin{aligned} V(t) &= \cos(u\,A(t)) \ , \\ W(t) &= -\tfrac{1}{2}\,\alpha(t)\,C(t) + \beta(t) \ , \\ Z(t) &= V(t)\ \exp\{W(t)\} \ , \end{aligned}$$

where $\alpha(t)$ and $\beta(t)$ are two continuously differentiable functions defined on $[0,\infty)$ with values in $\mathbb{R}$, to be specified later on.

(v) **Show that $V(t)$ and $W(t)$ are two Itô processes (and give their decompositions in terms of a usual integral and a stochastic integral).**

(vi) **Finally, show that $Z(t)$ is an Itô process (and give its decomposition in terms of a usual integral and a stochastic integral).**

Give ODE's that the functions $\alpha(t)$ and $\beta(t)$ should satisfy, for the usual integral vanish in the decomposition.

(vii) **Let $T > 0$ be fixed. Check that the two functions**

$$\alpha(t) = u\ \tanh((T-t)\,u) \qquad \textbf{and} \qquad \beta(t) = -\log\cosh((T-t)\,u) \ ,$$

satisfy the ODE's introduced in the answer to question (vi).

Here, 'tanh' denotes the hyperbolic tangent function.

(viii) **Let $\alpha(t)$ and $\beta(t)$ be defined as in question (vii). Check that the process $Z(t)$ is a square–integrable martingale in this case.**

(ix) **Conclude and check that**

$$\mathbb{E}[\exp\{i\,u\,A(t)\}] = \frac{1}{\cosh(u\,t)} \ . \qquad (\star)$$

holds for any real number u.