Random Models of Dynamical Systems
Introduction to SDE's (5/5)

4GM–AROM

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December 10, 2018

consider the simpler equation

$$X(t) = X(0) + \int_0^t b(X(s))\, ds + \int_0^t \sigma(X(s))\, dB(s)$$

with a m–dimensional Brownian motion $B = (B(t)\,,\, t \geqslant 0)$, and time–independent coefficients:

- a d–dimensional *drift* vector $b(x)$ defined on $\mathbb{R}^d$
- a $d \times m$ *diffusion* matrix $\sigma(x)$ defined on $\mathbb{R}^d$

global Lipschitz condition: there exists a positive constant $L > 0$ such that for any $x, x' \in \mathbb{R}^d$

$$|b(x) - b(x')| \leqslant L\,|x - x'| \qquad \text{and} \qquad \|\sigma(x) - \sigma(x')\| \leqslant L\,|x - x'|$$

linear growth condition (simple consequence in this case): there exists a positive constant $K > 0$ such that for any $x \in \mathbb{R}^d$

$$|b(x)| \leqslant K\,(1 + |x|) \qquad \text{and} \qquad \|\sigma(x)\| \leqslant K\,(1 + |x|)$$

Strong vs. weak error

objective: associated with a uniform subdivision $0 = t_0 < \cdots < t_k < \cdots$ (with constant time–step $h = t_k - t_{k-1}$), design a numerical scheme $\bar{X}_k$ that approximates the solution $X(t_k)$, and provide an approximate continuous–time process $\bar{X}(t)$ (to be made precise later on)

Definition the numerical scheme is *strongly* convergent of order $\alpha > 0$ if for any $0 \leqslant t \leqslant T$

$$\{\mathbb{E}|X(t) - \bar{X}(t)|^2\}^{1/2} \leqslant C(T)\, h^\alpha$$

Definition [approximation of moments] the numerical scheme is *weakly* convergent of order $\beta > 0$ if for any regular enough real–valued function f and for any $0 \leqslant t \leqslant T$

$$|\,\mathbb{E}[f(X(t))] - \mathbb{E}[f(\bar{X}(t))]\,| \leqslant C(f, T)\, h^\beta$$

Remark if a numerical scheme is strongly convergent of order $\alpha > 0$, then it is also weakly convergent of the same order $\alpha > 0$ (for a Lipschitz continuous function f)

indeed: if

$$|f(x) - f(x')| \leqslant L\,|x - x'|$$

for any $x, x' \in \mathbb{R}^d$, then

$$\begin{aligned}
|\,\mathbb{E}[f(X(t))] - \mathbb{E}[f(\bar{X}(t))]\,| &= |\,\mathbb{E}[f(X(t)) - f(\bar{X}(t))]\,| \\
&\leqslant \mathbb{E}|f(X(t)) - f(\bar{X}(t))| \\
&\leqslant L\,\mathbb{E}|X(t) - \bar{X}(t)| \\
&\leqslant L\,\{\mathbb{E}|X(t) - \bar{X}(t)|^2\}^{1/2} \\
&\leqslant L\,C(T)\,h^{\alpha}
\end{aligned}$$

Euler scheme

special important case: Euler scheme

same initial condition $\bar{X}_0 = X(0)$ for $k = 0$, and for any $k \geqslant 1$

$$\bar{X}_k = \bar{X}_{k-1} + b(\bar{X}_{k-1})\,(t_k - t_{k-1}) + \sigma(\bar{X}_{k-1})\,(B(t_k) - B(t_{k-1}))$$

and continuous–time approximation interpolating points $\bar{X}_k$ at time instants t_k

$$\bar{X}(t) = \bar{X}_{k-1} + b(\bar{X}_{k-1})\,(t - t_{k-1}) + \sigma(\bar{X}_{k-1})\,(B(t) - B(t_{k-1}))$$

for any time $t_{k-1} \leqslant t \leqslant t_k$ between two discretization times

Euler approximation seen as an Itô process, with frozen coefficients on each interval of the subdivision: indeed, for any $t_{k-1} \leqslant t \leqslant t_k$

$$\bar{X}(t) = \bar{X}_{k-1} + \int_{t_{k-1}}^{t} b(\bar{X}(\pi(s)))\, ds + \int_{t_{k-1}}^{t} \sigma(\bar{X}(\pi(s)))\, dB(s)$$

and more generally for any $t \geqslant 0$

$$\bar{X}(t) = \bar{X}(0) + \int_0^t b(\bar{X}(\pi(s)))\, ds + \int_0^t \sigma(\bar{X}(\pi(s)))\, dB(s)$$

where

$$\pi(s) = t_{k-1} \qquad \text{and} \qquad \bar{X}(\pi(s)) = \bar{X}_{k-1} \qquad \text{if } t_{k-1} \leqslant s < t_k$$

there exists a positive constant $M(T)$, independent of the time–step h, such that

$$\max_{0 \leqslant t \leqslant T} \mathbb{E}|\bar{X}(t)|^2 \leqslant M(T)$$

Stochastic differential equations

Euler scheme: strong error estimate

Euler scheme: weak error estimate

Euler scheme: strong error estimate

Theorem 1 the Euler scheme is strongly convergent of order $\frac{1}{2}$, i.e.

$$\max_{0\leqslant t\leqslant T} \mathbb{E}|X(t) - \bar{X}(t)|^2 = O(h)$$

Proof for any time $t_{k-1} \leqslant t \leqslant t_k$ between two discretization times, it holds

$$X(t) = X(t_{k-1}) + \int_{t_{k-1}}^{t} b(X(s))\, ds + \int_{t_{k-1}}^{t} \sigma(X(s))\, dB(s)$$

and (Euler approximation interpolating points $\bar{X}_k$ at time instants t_k)

$$\bar{X}(t) = \bar{X}_{k-1} + b(\bar{X}_{k-1})\,(t - t_{k-1}) + \sigma(\bar{X}_{k-1})\,(B(t) - B(t_{k-1}))$$

by difference, for any $t_{k-1} \leqslant t \leqslant t_k$

$$X(t) - \bar{X}(t) = X(t_{k-1}) - \bar{X}_{k-1} + \int_{t_{k-1}}^{t} [b(X(s)) - b(\bar{X}_{k-1})]\, ds + \int_{t_{k-1}}^{t} [\sigma(X(s)) - \sigma(\bar{X}_{k-1})]\, dB(s)$$

using the Itô formula yields

$$|X(t) - \bar{X}(t)|^2 = |X(t_{k-1}) - \bar{X}_{k-1}|^2$$

$$+ 2 \int_{t_{k-1}}^{t} (X(s) - \bar{X}(s))^* \left[b(X(s)) - b(\bar{X}_{k-1})\right] ds$$

$$+ 2 \int_{t_{k-1}}^{t} (X(s) - \bar{X}(s))^* \left[\sigma(X(s)) - \sigma(\bar{X}_{k-1})\right] dB(s)$$

$$+ \int_{t_{k-1}}^{t} \|\sigma(X(s)) - \sigma(\bar{X}_{k-1})\|^2 \, ds$$

using the bound $2\, u^* v \leqslant |u|^2 + |v|^2$, and taking expectation (assuming the stochastic integral is a (true, square–integrable) martingale), yields

$$\mathbb{E}|X(t) - \bar{X}(t)|^2 \leqslant \mathbb{E}|X(t_{k-1}) - \bar{X}_{k-1}|^2$$

$$+ \mathbb{E} \int_{t_{k-1}}^{t} |X(s) - \bar{X}(s)|^2\, ds$$

$$+ \mathbb{E} \int_{t_{k-1}}^{t} |b(X(s)) - b(\bar{X}_{k-1})|^2\, ds$$

$$+ \mathbb{E} \int_{t_{k-1}}^{t} \|\sigma(X(s)) - \sigma(\bar{X}_{k-1})\|^2\, ds$$

note that

$$|b(X(s)) - b(\bar{X}_{k-1})|$$

$$\leqslant |b(X(s)) - b(X(t_{k-1}))| + |b(X(t_{k-1})) - b(\bar{X}_{k-1})|$$

$$\leqslant L \, [|X(s) - X(t_{k-1})| + |X(t_{k-1}) - \bar{X}_{k-1}|]$$

and similarly

$$\|\sigma(X(s)) - \sigma(\bar{X}_{k-1})\|$$

$$\leqslant L \, [|X(s) - X(t_{k-1})| + |X(t_{k-1}) - \bar{X}_{k-1}|]$$

with two different contributions to the error

- discretization error at previous iteration
- modulus of continuity of the solution

therefore

$$\mathbb{E}|X(t) - \bar{X}(t)|^2 \leqslant (1 + 4\,L^2\,(t - t_{k-1}))\,\mathbb{E}|X(t_{k-1}) - \bar{X}_{k-1}|^2$$

$$+ 4\,L^2\,\mathbb{E}\int_{t_{k-1}}^{t} |X(s) - X(t_{k-1})|^2\,ds$$

$$+ \mathbb{E}\int_{t_{k-1}}^{t} |X(s) - \bar{X}(s)|^2\,ds$$

note that the modulus of continuity for the solution satisfies

$$\mathbb{E}|X(s) - X(t_{k-1})|^2 \;\leqslant C\,(s - t_{k-1})$$

hence

$$\mathbb{E}|X(t) - \bar{X}(t)|^2 \;\leqslant\; (1 + 4\,L^2\,h)\,\mathbb{E}|X(t_{k-1}) - \bar{X}_{k-1}|^2 + 4\,L^2\,C\,h^2$$

$$+ \mathbb{E}\int_{t_{k-1}}^{t} |X(s) - \bar{X}(s)|^2\,ds$$

the Gronwall lemma yields

$$\mathbb{E}|X(t) - \bar{X}(t)|^2 \leqslant$$

$$\leqslant \left[(1 + 4\,L^2\,h)\,\mathbb{E}|X(t_{k-1}) - \bar{X}_{k-1}|^2 + 4\,L^2\,C\,h^2 \right]\ \exp\{t - t_{k-1}\}$$

introducing

$$\varepsilon_k = \max_{t_{k-1} \leqslant t \leqslant t_k} \mathbb{E}|X(t) - \bar{X}(t)|^2$$

it holds

$$\varepsilon_k \leqslant (1 + 4\,L^2\,h)\ \exp\{h\}\ \varepsilon_{k-1} + 4\,L^2\,C\,h^2\ \exp\{h\}$$

and by induction

$$\varepsilon_k \leqslant \frac{4\,L^2\,C\,h^2\ \exp\{h\}}{(1 + 4\,L^2\,h)\ \exp\{h\} - 1}\ \left[(1 + 4\,L^2\,h)\ \exp\{h\} \right]^k$$

note that

$$\frac{4\,L^2\,C\,h^2\ \exp\{h\}}{(1 + 4\,L^2\,h)\ \exp\{h\} - 1} = \frac{4\,L^2\,C\,h^2}{4\,L^2\,h + (1 - \exp\{-h\})} = O(h)$$

for any $k = 1 \cdots \lfloor T/h \rfloor$, the following bound holds

$$[(1 + 4\,L^2\,h)\,\exp\{h\}]^k \leqslant [(1 + 4\,L^2\,h)\,\exp\{h\}]^{\lfloor T/h \rfloor} \leqslant \exp\{(1 + 4\,L^2)\,T\}$$

therefore

$$\max_{0 \leqslant t \leqslant T} \mathbb{E}|X(t) - \bar{X}(t)|^2 = \max_{k=1 \cdots \lfloor T/h \rfloor} \varepsilon_k$$

$$\leqslant \frac{4\,L^2\,C\,h^2}{4\,L^2\,h + (1 - \exp\{-h\})}\ \exp\{(1 + 4\,L^2)\,T\} = O(h) \qquad \square$$

Stochastic differential equations

Euler scheme: strong error estimate

Euler scheme: weak error estimate

Euler scheme: weak error estimate

let $T \geqslant 0$ be fixed (as in the PDE) and consider specifically a uniform subdivision of the form $0 = t_0 < \cdots < t_k < \cdots < t_n = T$ of the interval $[0, T]$ (with constant time–step $h = T/n$)

Theorem 2 under some additional technical assumptions (on the coefficients of the SDE and on the test function) the Euler scheme is weakly convergent of order 1, i.e.

$$|\,\mathbb{E}[f(X(T))] - \mathbb{E}[f(\bar{X}(T))]\,| = O(h)$$

even more

$$\mathbb{E}[f(X(T))] - \mathbb{E}[f(\bar{X}(T))] = C(f, T)\; h + O(h^2)$$

Corollary 3 [Romberg-Richardson extrapolation] let $\bar{X}^h$ and $\bar{X}^{\frac{1}{2}h}$ be the Euler approximation with time–step h and $\frac{1}{2}\,h$ respectively, and define a further approximation as

$$f^{h,\frac{1}{2}h}(T) = 2\,\mathbb{E}[f(\bar{X}^{\frac{1}{2}h}(T))] - \mathbb{E}[f(\bar{X}^h(T))]$$

then

$$|\,\mathbb{E}[f(X(T))] - f^{h,\frac{1}{2}h}(T)\,| = O(h^2)$$

Proof indeed

$$\mathbb{E}[f(X(T))] - f^{h,\frac{1}{2}h}(T) = 2\,(\,\mathbb{E}[f(X(T))] - \mathbb{E}[f(\bar{X}^{\frac{1}{2}h}(T))]\,)$$

$$-\,(\,\mathbb{E}[f(X(T))] - \mathbb{E}[f(\bar{X}^h(T))]\,)$$

$$= 2\,[C(f,T)\,\tfrac{1}{2}\,h + O(h^2)] - [C(f,T)\,h + O(h^2)] = O(h^2) \qquad \square$$

Corollary 4 [Monte Carlo approximation] let $(\bar{X}^{h,i}\,,\ i = 1 \cdots N)$ be N independent realizations of the same Euler scheme with step–size h, and consider the empirical mean

$$\widehat{f}^{h,N}(T) = \frac{1}{N}\,\sum_{i=1}^{N} f(\bar{X}^{h,i}(T))$$

as a (random) practical approximation, then (bias^2 + variance)

$$\mathbb{E}\,|\widehat{f}^{h,N}(T) - \mathbb{E}[f(X(T))]|^2 = C^2(f,T)\,h^2 + \frac{\mathrm{var}(f(\bar{X}^h(T)))}{N} + O(h^3)$$

Proof clearly

$$\mathbb{E}[\widehat{f}^{h,N}(T)] = \mathbb{E}[f(\bar{X}^h(T))]$$

hence

$$\mathbb{E}[\widehat{f}^{h,N}(T)] - \mathbb{E}[f(X(T))] = C(f,T)\,h + O(h^2)$$

and

$$\mathbb{E}\,|\widehat{f}^{h,N}(T) - \mathbb{E}[f(\bar{X}^h(T))]\,|^2 = \frac{\mathrm{var}(f(\bar{X}^h(T)))}{N} \qquad \square$$

with the solution of the SDE is associated the second–order partial differential operator

$$L = \sum_{i=1}^{d} b_i(\cdot) \, \frac{\partial}{\partial x_i} + \tfrac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(\cdot) \, \frac{\partial^2}{\partial x_i \, \partial x_j}$$

let $u(t,x)$ be the unique (and 'regular enough') solution of the PDE (running backward from T to 0)

$$\frac{\partial u}{\partial t}(t,x) + L\,u(t,x) = 0 \qquad \text{for any } (t,x) \text{ in } [0,T] \times \mathbb{R}^d$$

$$u(T,x) = f(x) \qquad \text{for any } x \text{ in } \mathbb{R}^d$$

Theorem 5 * if the drift b and the diffusion matrix $a = \sigma\sigma^*$ have C^∞ regularity, with bounded derivatives of any order, if the test–function f has C^∞ regularity, and at most polynomial growth, then there exists a unique solution $u(t, x)$ to the PDE and this solution has also C^∞ regularity and at most polynomial growth

Remark this PDE is just instrumental in the proof, i.e. the numerical scheme does not use the solution $u(t, x)$ explicitly

Proof of Theorem 2 recall that the Itô formula yields

$$u(t, X(t)) = u(s, X(s)) + \int_s^t [\frac{\partial u}{\partial t}(r, X(r)) + L\, u(r, X(r))]\, dr$$

$$+ \int_s^t u'(r, X(r))\, \sigma(X(r))\, dB(r)$$

and under the assumptions, the stochastic integral is a (true, square–integrable) martingale, hence

$$\mathbb{E}[u(t, X(t))] = \mathbb{E}[u(s, X(s))]$$

note that

$$\mathbb{E}[f(X(T))] = \mathbb{E}[u(T, X(T))] = \mathbb{E}[u(0, X(0))] = \mathbb{E}[u(0, \bar{X}_0)]$$

and $f(\bar{X}(T)) = u(T, \bar{X}_n)$ (initial condition at time $T = t_n$), hence

$$\mathbb{E}[f(\bar{X}(T))] - \mathbb{E}[f(X(T))] = \mathbb{E}[u(T, \bar{X}_n) - u(0, \bar{X}_0)]$$

$$= \sum_{k=1}^{n} \mathbb{E}[u(t_k, \bar{X}_k) - u(t_{k-1}, \bar{X}_{k-1})]$$

with the Euler approximation

$$\bar{X}(t) = \bar{X}_{k-1} + b(\bar{X}_{k-1})\,(t - t_{k-1})$$

$$+ \sigma(\bar{X}_{k-1})\,(B(t) - B(t_{k-1}))$$

valid for $t_{k-1} \leqslant t \leqslant t_k$, is associated the second–order partial differential operator with constant coefficients

$$L_k = \sum_{i=1}^{d} b_i(\bar{X}_{k-1})\,\frac{\partial}{\partial x_i} + \tfrac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(\bar{X}_{k-1})\,\frac{\partial^2}{\partial x_i\,\partial x_j}$$

note that

$$L_k\,\phi(x) = \sum_{i=1}^{d} b_i(\bar{X}_{k-1})\,\frac{\partial \phi}{\partial x_i}(x) + \tfrac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(\bar{X}_{k-1})\,\frac{\partial^2 \phi}{\partial x_i\,\partial x_j}(x)$$

and

$$L\,\phi(x) = \sum_{i=1}^{d} b_i(x)\,\frac{\partial \phi}{\partial x_i}(x) + \tfrac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(x)\,\frac{\partial^2 \phi}{\partial x_i\,\partial x_j}(x)$$

coincide when $x = \bar{X}_{k-1}$

using the Itô formula for the Euler approximation and for the time–dependent function $u(t,x)$ yields

$$u(t,\bar{X}(t)) - u(t_{k-1},\bar{X}_{k-1}) =$$

$$= \int_{t_{k-1}}^{t} [\frac{\partial u}{\partial t}(s,\bar{X}(s)) + L_k\, u(s,\bar{X}(s))]\, ds + \int_{t_{k-1}}^{t} u'(s,\bar{X}(s))\, \sigma(\bar{X}_{k-1})\, dB(s)$$

$$= \int_{t_{k-1}}^{t} [\frac{\partial u}{\partial t}(s,\bar{X}(s)) + L\, u(s,\bar{X}(s))]\, ds + \int_{t_{k-1}}^{t} u'(s,\bar{X}(s))\, \sigma(\bar{X}_{k-1})\, dB(s)$$

$$+ \int_{t_{k-1}}^{t} [L_k\, u(s,\bar{X}(s)) - L\, u(s,\bar{X}(s))]\, ds$$

$$= \int_{t_{k-1}}^{t} [L_k\, u(s,\bar{X}(s)) - L\, u(s,\bar{X}(s))]\, ds + \int_{t_{k-1}}^{t} u'(s,\bar{X}(s))\, \sigma(\bar{X}_{k-1})\, dB(s)$$

since

$$\frac{\partial u}{\partial t}(s,y) + L\, u(s,y) = 0$$

for any $y \in \mathbb{R}^d$, and the identity holds in particular for $y = \bar{X}(s)$

taking expectation (assuming that the stochastic integral has zero expectation) yields

$$\mathbb{E}[u(t,\bar{X}(t)) - u(t_{k-1},\bar{X}_{k-1})]$$

$$= \mathbb{E}\int_{t_{k-1}}^{t} [L_k\, u(s,\bar{X}(s)) - L\, u(s,\bar{X}(s))]\, ds$$

$$= \mathbb{E}\int_{t_{k-1}}^{t} u'(s,\bar{X}(s))\, [b(\bar{X}_{k-1}) - b(\bar{X}(s))]\, ds$$

$$+ \tfrac{1}{2}\, \mathbb{E}\int_{t_{k-1}}^{t} \mathrm{trace}[u''(s,\bar{X}(s))\, [a(\bar{X}_{k-1}) - a(\bar{X}(s))]\,]\, ds$$

the next step is to write the Itô formula for the time–dependent functions

$$v(s,x) \;=\; u'(s,x)\, (b(\bar{X}_{k-1}) - b(x))$$

$$v(s,x) \;=\; \tfrac{1}{2}\,\mathrm{trace}[u''(s,x)\, (a(\bar{X}_{k-1}) - a(x))\,]$$

this requires some regularity

note that $v(t_{k-1}, \bar{X}_{k-1}) = 0$ in both cases, hence

$$v(s, \bar{X}(s)) = \int_{t_{k-1}}^{s} [\frac{\partial v}{\partial t}(r, \bar{X}(r)) + L_k\, v(r, \bar{X}(r))]\, dr + \int_{t_{k-1}}^{s} v'(r, \bar{X}(r))\, \sigma(\bar{X}_{k-1})\, dB(r)$$

then

$$\int_{t_{k-1}}^{t} v(s, \bar{X}(s))\, ds = \int_{t_{k-1}}^{t} \int_{t_{k-1}}^{s} [\frac{\partial v}{\partial t}(r, \bar{X}(r)) + L_k\, v(r, \bar{X}(r))]\, dr\, ds + \int_{t_{k-1}}^{t} \int_{t_{k-1}}^{s} v'(r, \bar{X}(r))\, \sigma(\bar{X}_{k-1})\, dB(r)\, ds$$

taking expectation (assuming that the stochastic integral has zero expectation) yields

$$\mathbb{E} \int_{t_{k-1}}^{t} v(s, \bar{X}(s))\, ds = \mathbb{E} \int_{t_{k-1}}^{t} \int_{t_{k-1}}^{s} [\frac{\partial v}{\partial t}(r, \bar{X}(r)) + L_k\, v(r, \bar{X}(r))]\, dr\, ds = O((t - t_{k-1})^2)$$

[indeed, introducing

$$\psi(s) = \int_0^s \phi(r)\, dB(r)$$

and using the integration by parts formula yields

$$t\,\psi(t) = \int_0^t \psi(s)\, ds + \int_0^t s\,\phi(s)\, dB(s)$$

hence

$$\int_0^t \Big[\int_0^s \phi(r)\, dB(r)\Big]\, ds \; = \; t \int_0^t \phi(s)\, dB(s) - \int_0^t s\,\phi(s)\, dB(s)$$

$$= \int_0^t (t-s)\,\phi(s)\, dB(s)$$

is expressed as a stochastic integral]

under some regularity assumptions on

- the coefficients (drift vector and diffusion matrix) of the stochastic differential equation
- the solution of the partial differential equation

the following estimate holds

$$\mathbb{E}[u(t_k, \bar{X}_{k-1}) - u(t_{k-1}, \bar{X}_{k-1})] = O((t_k - t_{k-1})^2)$$

i.e.

$$|\, \mathbb{E}[u(t_k, \bar{X}_{k-1}) - u(t_{k-1}, \bar{X}_{k-1})] \,| \leqslant C\, (t_k - t_{k-1})^2$$

where the constant C does not depend on the time–step, hence

$$|\, \mathbb{E}[f(\bar{X}(T))] - \mathbb{E}[f(X(T))] \,| \leqslant \sum_{k=1}^{n} |\, \mathbb{E}[u(t_k, \bar{X}_k) - u(t_{k-1}, \bar{X}_{k-1})] \,|$$

$$\leqslant C \sum_{k=1}^{n} (t_k - t_{k-1})^2 \leqslant C\, T\, h \qquad \square$$

Remark if the test–function f is not regular, for instance some indicator function, then assumptions of the theorem are not satisfied

provided the drift b and the diffusion matrix a have the same regularity and growth condition as in the theorem, and if the following *uniform ellipticity* (non–degeneracy) condition holds

$$v^* \, a(x) \, v \geqslant \lambda \, |v|^2$$

for any $x \in \mathbb{R}^d$ and any d–dimensional vector v, then the properties (weak convergence of order 1 and expansion of the error) remain true

the proof relies on Malliavin calculus (or stochastic calculus of variations) and is far beyond the scope of this course

Illustration #1

Brownian motion on the circle

$$X(t) = X(0) - \int_0^t F\,X(s)\,ds + \int_0^t R\,X(s)\,dB(s)$$

with initial condition $X(0) = (0, 1)$, and with 2×2 matrices

$$F = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

time–invariant: $|X(t)| = 1$ for any $t \geqslant 0$

objective: approximate time–invariant $\mathbb{E}|X(t)|^2 = 1$ for any $t \geqslant 0$, using

- Euler approximation with time–step h or $\frac{1}{2}h$
- Monte Carlo approximation with N samples

i.e. coarse grid approximation

$$\widehat{f}^{h,N}(t) = \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^{h,i}(t))|^2$$

fine grid approximation

$$\widehat{f}^{\frac{1}{2}h,N}(t) = \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^{\frac{1}{2}h,i}(t))|^2$$

and Romberg–Richardson extrapolation

$$\widehat{f}^{h,\frac{1}{2}h,N}(t) = 2\,\widehat{f}^{\frac{1}{2}h,N}(t) - \widehat{f}^{h,N}(t)$$

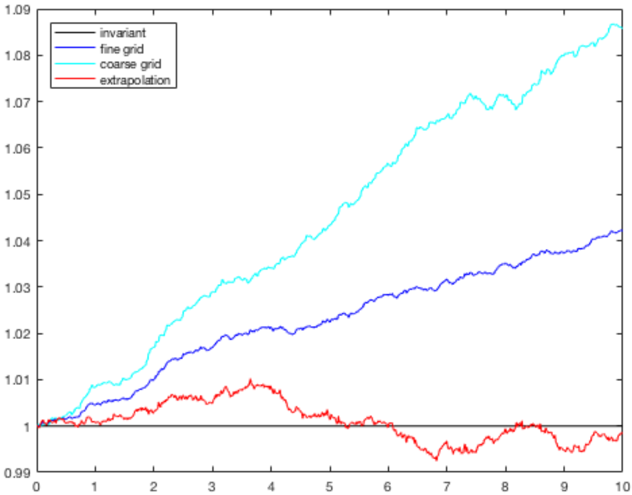

1.09
1.08
1.07
1.06
1.05
1.04
1.03
1.02
1.01
1
0.99
0
1
2
3
4
5
6
7
8
9
10
invariant
fine grid
coarse grid
extrapolation

Illustration #2

two–dimensional stationary Gaussian diffusion

$$X(t) = X(0) + \int_0^t (-\tfrac{1}{2}\, I + R)\, X(s)\, ds + B(t)$$

with initial condition $X(0) \sim \mathcal{N}(0, I)$, and with 2×2 matrice

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

time–invariant (in distribution): $X(t) \sim \mathcal{N}(0, I)$, in particular $\mathbb{E}[X(t)\, X^*(t)] = I$, for any $t \geqslant 0$

objective: approximate time–invariant $\mathbb{E}|X(t)|^2 = 2$ for any $t \geqslant 0$, using

- Euler approximation with time–step h or $\frac{1}{2}h$
- Monte Carlo approximation with N samples

i.e. coarse grid approximation

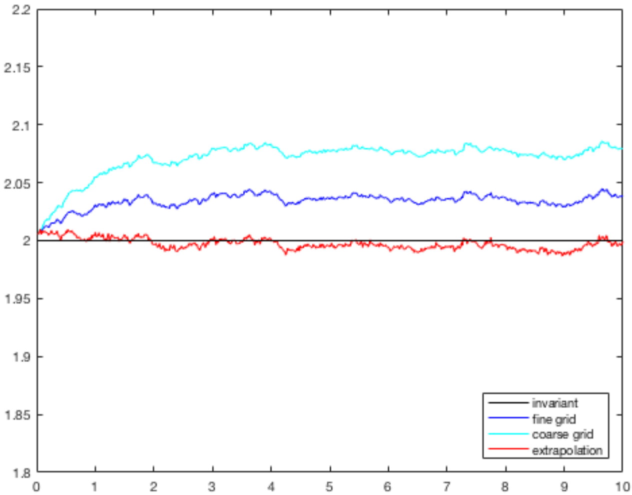
$$\widehat{f}^{h,N}(t) = \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^{h,i}(t))|^2$$

fine grid approximation

$$\widehat{f}^{\frac{1}{2}h,N}(t) = \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^{\frac{1}{2}h,i}(t))|^2$$

and Romberg–Richardson extrapolation

$$\widehat{f}^{h,\frac{1}{2}h,N}(t) = 2\,\widehat{f}^{\frac{1}{2}h,N}(t) - \widehat{f}^{h,N}(t)$$


2.2
2.15
2.1
2.05
2
1.95
1.9
1.85
1.8
0
1
2
3
4
5
6
7
8
9
10
invariant
fine grid
coarse grid
extrapolation