INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems
Introduction to SDE's

Written Exam (aka DS)

January 8, 2019

Stochastic integral in intrinsic clock = Brownian motion

Let B be a one–dimensional standard Brownian motion, with $B(0) = 0$, and adapted to a given filtration $\mathcal{F}$, and consider the stochastic process X defined by

$$X(t) = \int_0^t \phi(u)\, dB(u) \ ,$$

for any $t \geq 0$, where ϕ belongs to M^2_{loc}, i.e.

$$A(t) = \int_0^t |\phi(u)|^2\, du < \infty \ ,$$

almost surely, for any $t \geq 0$.

(i) **Write the Itô formula for the process X and for the complex–valued function $f(x) = \exp\{i\,\lambda\, x\}$ where the scalar λ is fixed, between the time instants s and t, with $0 \leq s \leq t$.**

Solution

Clearly, the complex–valued function $f(x) = \exp\{i\,\lambda\, x\}$ is twice continuously differentiable, with $f'(x) = i\,\lambda\, f(x)$ and $f''(x) = -\lambda^2\, f(x)$, and writing the Itô formula yields

$$\begin{aligned} \exp\{i\,\lambda\, X(t)\} \ &= \ \exp\{i\,\lambda\, X(s)\} + i\,\lambda \int_s^t \exp\{i\,\lambda\, X(u)\}\, \phi(u)\, dB(u) \\ &\quad - \tfrac{1}{2}\,\lambda^2 \int_s^t \exp\{i\,\lambda\, X(u)\}\, |\phi(u)|^2\, du \ , \end{aligned}$$

for any $0 \leq s \leq t$.

□

For any $t \geq 0$, define

$$\tau(t) = \inf\{s \geq 0 \ : \ \int_0^s |\phi(u)|^2\, du \geq t\} \ ,$$

if such time exists, and $\tau(t) = \infty$ otherwise.

(ii) **For any $t \geq 0$, show that the random variable $\tau(t)$ is a stopping time, and that the equivalent definition**

$$\tau(t) = \inf\{s \geq 0 \,:\, \int_0^s |\phi(u)|^2 \, du = t\} \ .$$

holds, hence $A(\tau(t)) = t$.

Show that $\tau(t) \uparrow \infty$ almost surely as $t \uparrow \infty$.

Solution

Note that the mapping $t \mapsto A(t)$ is continuous, hence the first time when any level is exceeded is also the first time when the same level is reached.

Note that the mapping $t \mapsto A(t)$ is non–decreasing, and two cases can occur: either $A(t)$ has a finite limit or $A(t)$ has an infinite limit as $t \uparrow \infty$. In the first case where $A(t)$ has a finite limit as $t \uparrow \infty$, then for any level t above this limit there is no such time $s \geq 0$ such that $A(s) = t$ hence $\tau(t) = \infty$. In the second case where $A(t)$ has an infinite limit as $t \uparrow \infty$, then $\tau(t)$ is finite for any finite time t, and (because the mapping $t \mapsto \tau(t)$ is non–decreasing) this implies that $\tau(t)$ has an infinite limit as $t \uparrow \infty$: indeed, if $\tau(t)$ would have a finite limit as $t \uparrow \infty$, then for any u above this limit the level $A(u)$ could not be reached in finite time, which is a contradiction. In both cases it holds that $\tau(t) \uparrow \infty$ as $t \uparrow \infty$.

□

From now on, it is assumed that

$$\int_0^\infty |\phi(u)|^2 \, du = \infty \ ,$$

i.e. $A(t) \uparrow \infty$ almost surely as $t \uparrow \infty$, so that $\tau(t) < \infty$ for any $t < \infty$.

(iii) **Show (a simple graphic could help to prove (a) and (b)) that**

(a) the mapping $t \mapsto \tau(t)$ is non–decreasing and left–continuous,

(b) for any $t, s \geq 0$

$$\{A(t) \geq s\} = \{\tau(s) \leq t\} \ ,$$

(c) for any nonnegative Borel measurable function f and for any $t \geq 0$

$$\int_0^t f(\tau(s)) \, ds = \int_0^{\tau(t)} f(s) \, dA(s) \ .$$

[Hint: just prove (c) for any function f of the form of an indicator function, defined by $f(s) = 1_{(0 \,\leq\, s \,\leq\, L)}$ for any $s \geq 0$ and for some $L > 0$ (the same result for an arbitrary nonnegative Borel function would follow by a monotone class argument).]

Solution

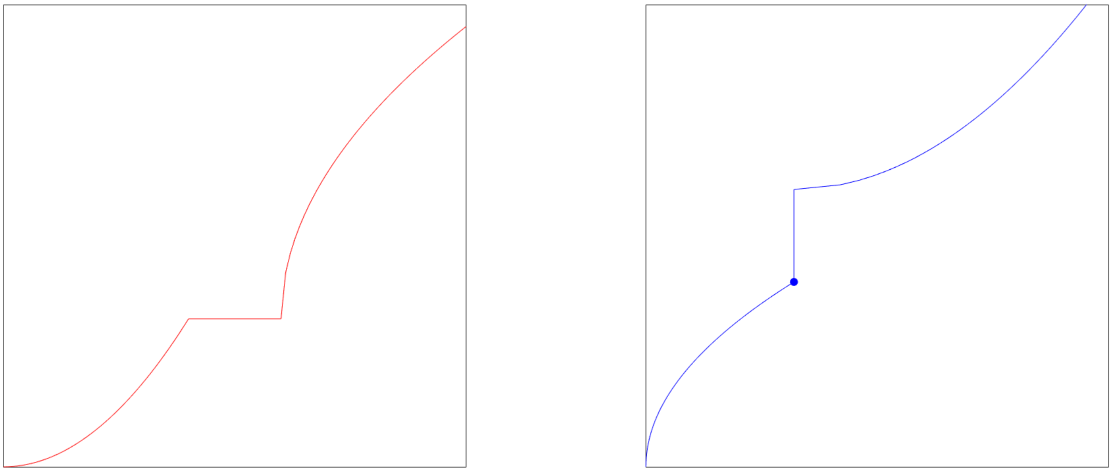

Figure 1: increasing process $t \mapsto A(t)$ vs. intrinsic clock $t \mapsto \tau(t)$

Let f be an indicator function, defined by $f(s) = 1_{(0 \leq s \leq L)}$ for any $s \geq 0$ and for some $L > 0$, and let $t \geq 0$ be fixed. Firstly

$$\int_0^{\tau(t)} f(s)\, dA(s) = \int_0^{\tau(t)} 1_{(0 \leq s \leq L)}\, dA(s) = \int_0^{\tau(t)\wedge L} dA(s) = A(\tau(t) \wedge L) ,$$

and secondly, using point (b) yields

$$\int_0^t f(\tau(s))\, ds = \int_0^t 1_{(0 \leq \tau(s) \leq L)}\, ds = \int_0^t 1_{(0 \leq s \leq A(L))}\, ds = A(L) \wedge t .$$

Two cases can occur: either $\tau(t) > L$ or $\tau(t) \leq L$. In the first case where $\tau(t) > L$, then firstly $A(\tau(t) \wedge L) = A(L)$, and secondly using point (b) yields $A(L) < t$ hence $A(L) \wedge t = A(L)$. In the second case where $\tau(t) \leq L$, then firstly $A(\tau(t) \wedge L) = A(\tau(t)) = t$, and secondly using point (b) yields $A(L) \geq t$ hence $A(L) \wedge t = t$. In both cases, it holds $A(\tau(t) \wedge L) = A(L) \wedge t$, hence

$$\int_0^{\tau(t)} f(s)\, dA(s) = \int_0^t f(\tau(s))\, ds .$$

$\square$

The mapping τ is called the *intrinsic clock* (or *intrinsic time*) for the stochastic process X, and the time–changed stochastic process Z is defined by

$$Z(t) = X(\tau(t)) = \int_0^{\tau(t)} \phi(u)\, dB(u) ,$$

for any $t \geq 0$.

(iv) **Using the representation obtained in question (i) and using the result obtained in question (iii-c), show that**

$$\exp\{i\,\lambda\,Z(t)\} \;=\; \exp\{i\,\lambda\,Z(s)\} + i\,\lambda \int_{\tau(s)}^{\tau(t)} \exp\{i\,\lambda\,X(u)\}\,\phi(u)\,dB(u)$$

$$- \tfrac{1}{2}\,\lambda^2 \int_s^t \exp\{i\,\lambda\,Z(u)\}\,du \;,$$

for any $0 \leq s \leq t$**.**

SOLUTION

Note that $X(\tau(s)) = Z(s)$ and $X(\tau(t)) = Z(t)$ and using the representation obtained in question (i) between times $\tau(s)$ and $\tau(t)$ yields

$$\exp\{i\,\lambda\,Z(t)\} \;=\; \exp\{i\,\lambda\,Z(s)\} + i\,\lambda \int_{\tau(s)}^{\tau(t)} \exp\{i\,\lambda\,X(u)\}\,\phi(u)\,dB(u)$$

$$- \tfrac{1}{2}\,\lambda^2 \int_{\tau(s)}^{\tau(t)} \exp\{i\,\lambda\,X(u)\}\,|\phi(u)|^2\,du \;,$$

for any $0 \leq s \leq t$. Using the result obtained in question (iii-c) yields

$$\int_0^{\tau(t)} \exp\{i\,\lambda\,X(u)\}\,|\phi(u)|^2\,du \;=\; \int_0^{\tau(t)} \exp\{i\,\lambda\,X(u)\}\,dA(u)$$

$$= \int_0^t \exp\{i\,\lambda\,X(\tau(u))\}\,du = \int_0^t \exp\{i\,\lambda\,Z(u)\}\,du \;,$$

and by difference

$$\int_{\tau(s)}^{\tau(t)} \exp\{i\,\lambda\,X(u)\}\,|\phi(u)|^2\,du = \int_s^t \exp\{i\,\lambda\,Z(u)\}\,du \;.$$

□

Introduce the σ–algebra $\mathcal{A}(t) = \mathcal{F}(\tau(t))$, i.e. $A \in \mathcal{A}(t)$ iff for any $u \geq 0$

$$A \cap \{\tau(t) \leq u\} \in \mathcal{F}(u) \;.$$

If the stochastic process M, defined by the stochastic integral

$$M(t) = \int_0^t \exp\{i\,\lambda\,X(u)\}\,\phi(u)\,dB(u) \;,$$

for any $t \geq 0$, would be a martingale, and if the stopping time $\tau(t)$ would be almost surely bounded, then the optional sampling theorem would yield

$$\mathbb{E}[M(\tau(t)) - M(\tau(s)) \mid \mathcal{A}(s)] = \mathbb{E}[\int_{\tau(s)}^{\tau(t)} \exp\{i\,\lambda\,X(u)\}\,\phi(u)\,dB(u) \mid \mathcal{A}(s)] = 0 \;,$$

for any $0 \leq s \leq t$. The purpose of the next question is to show that the same identity holds in the more general case considered here, where ϕ does belong to $M^2_{\rm loc}$ only.

(v) **Show that the stopped process $M^{\tau(t_2)}$, defined by $M^{\tau(t_2)}(t) = M(t \wedge \tau(t_2))$ for any $t \geq 0$, is a square–integrable martingale, and that**

$$\mathbb{E}[\int_{\tau(t_1)}^{\tau(t_2)} \exp\{i\,\lambda\,X(u)\}\,\phi(u)\,dB(u) \mid \mathcal{A}(t_1)] = 0 \ ,$$

for any $0 \leq t_1 \leq t_2$.

[Hint: recall that for a uniformly integrable martingale, the optional sampling theorem holds for any almost surely *finite* (and not necessarily *bounded*) stopping times.]

Solution

Let $t_2 \geq 0$ be fixed. Following the same approach as for the extension of stochastic integral by *localization*, introduce the stochastic process $M^{\tau(t_2)}$ defined by

$$M^{\tau(t_2)}(t) = \int_0^t 1_{(0 \,\leq\, u \,\leq\, \tau(t_2))} \ \exp\{i\,\lambda\,X(u)\}\,\phi(u)\,dB(u) \ ,$$

for any $t \geq 0$. Although the integrand $u \mapsto \exp\{i\,\lambda\,X(u)\}\,\phi(u)$ does belong to M^2_{loc} only and not necessarily to $M^2([0,\infty))$, the integrand $u \mapsto 1_{(0 \,\leq\, u \,\leq\, \tau(t_2))} \ \exp\{i\,\lambda\,X(u)\}\,\phi(u)$ does belong to $M^2([0,\infty))$: indeed

$$\mathbb{E} \int_0^\infty |1_{(0 \,\leq\, u \,\leq\, \tau(t_2))} \ \exp\{i\,\lambda\,X(u)\}\,\phi(u)|^2\,du \leq \mathbb{E} \int_0^{\tau(t_2)} |\phi(u)|^2\,du \leq t_2 < \infty \ .$$

Therefore, the stochastic process $M^{\tau(t_2)}$ is a square–integrable martingale, hence a *uniformly integrable* martingale. Note that uniform integrability could be obtained directly, since

$$\begin{aligned} \mathbb{E}|M^{\tau(t_2)}(t)|^2 \ &= \ \mathbb{E} \int_0^t |1_{(0 \,\leq\, u \,\leq\, \tau(t_2))} \ \exp\{i\,\lambda\,X(u)\}\,\phi(u)|^2\,du \\ &\leq \ \mathbb{E} \int_0^{\tau(t_2)} |\phi(u)|^2\,du \leq t_2 < \infty \ , \end{aligned}$$

hence

$$\sup_{t \geq 0} \ \mathbb{E}|M^{\tau(t_2)}(t)|^2 \leq t_2 < \infty \ .$$

The optional sampling theorem for a uniformly integrable martingale holds for any almost surely *finite* (and not necessarily *bounded*) stopping times, such as $0 \leq \tau(t_1) \leq \tau(t_2) < \infty$, hence

$$\mathbb{E}[M^{\tau(t_2)}(\tau(t_2)) \mid \mathcal{F}(\tau(t_1))] = M^{\tau(t_2)}(\tau(t_1)) \ ,$$

or in other words

$$\mathbb{E}[M(\tau(t_2)) \mid \mathcal{A}(t_1)] = M(\tau(t_1)) \ ,$$

hence

$$\mathbb{E}[M(\tau(t_2)) - M(\tau(t_1)) \mid \mathcal{A}(t_1)] = \mathbb{E}[\int_{\tau(t_1)}^{\tau(t_2)} \exp\{i\,\lambda\,X(u)\}\,\phi(u)\,dB(u) \mid \mathcal{A}(t_1)] = 0 \ .$$

□

(vi) **Show that the following expression**

$$\mathbb{E}[\,\exp\{i\,\lambda\,(Z(t)-Z(s))\}\mid \mathcal{A}(s)] = \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\}\ ,$$

holds for the conditional characteristic function.

Conclude that the process Z is a standard Brownian motion w.r.t. the filtration $\mathcal{A} = (\mathcal{A}(t)\,,\,t \geq 0)$.

Solution

Taking conditional expectation w.r.t. $\mathcal{A}(s)$ in the representation obtained in question (iv) and using the result obtained in question (v) yields

$$\begin{aligned}\mathbb{E}[\exp\{i\,\lambda\,Z(t)\}\mid \mathcal{A}(s)] \;&=\; \exp\{i\,\lambda\,Z(s)\} + i\,\lambda\,\mathbb{E}[\int_{\tau(s)}^{\tau(t)} \exp\{i\,\lambda\,X(u)\}\,\phi(u)\,dB(u)\mid \mathcal{A}(s)]\\ &\quad -\tfrac{1}{2}\,\lambda^2\,\mathbb{E}[\int_s^t \exp\{i\,\lambda\,Z(u)\}\,du \mid \mathcal{A}(s)]\\ &=\; \exp\{i\,\lambda\,Z(s)\} - \tfrac{1}{2}\,\lambda^2\,\mathbb{E}[\int_s^t \exp\{i\,\lambda\,Z(u)\}\,du \mid \mathcal{A}(s)]\ ,\end{aligned}$$

or equivalently

$$\begin{aligned}\mathbb{E}[\exp\{i\,\lambda\,(Z(t)-Z(s))\}\mid \mathcal{A}(s)] \;&=\; 1 - \tfrac{1}{2}\,\lambda^2\,\mathbb{E}[\int_s^t \exp\{i\,\lambda\,(Z(u)-Z(s))\}\,du \mid \mathcal{A}(s)]\\ &=\; 1 - \tfrac{1}{2}\,\lambda^2 \int_s^t \mathbb{E}[\exp\{i\,\lambda\,(Z(u)-Z(s))\}\mid \mathcal{A}(s)]\,du\ ,\end{aligned}$$

for any $t \geq s$. Therefore, the function defined by

$$V(t) = \mathbb{E}[\exp\{i\,\lambda\,(Z(t)-Z(s))\}\mid \mathcal{A}(s)]\ ,$$

for any $t \geq s$, satisfies the ordinary differential equation

$$V(t) = 1 - \tfrac{1}{2}\,\lambda^2 \int_s^t V(u)\,du\ ,$$

with explicit solution

$$V(t) = \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\}\ ,$$

for any $t \geq s$. The identity

$$\mathbb{E}[\exp\{i\,\lambda\,(Z(t)-Z(s))\}\mid \mathcal{A}(s)] = \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\}\ ,$$

valid for any scalar λ, shows that the increment $(Z(t)-Z(s))$ is a Gaussian random variable, independent of $\mathcal{A}(s)$, with mean zero and variance $(t-s)$, for any $t \geq s$, hence the time–changed stochastic process Z is a Brownian motion adapted to the filtration $\mathcal{A}$.

This result is conveniently summarized by the statement:

stochastic integral in intrinsic clock = Brownian motion.

□

(vii) **Show that**

$$\frac{\displaystyle\int_0^T \phi(u)\,dB(u)}{\displaystyle\int_0^T |\phi(u)|^2\,du} \longrightarrow 0\ ,$$

almost surely as $T \uparrow \infty$.

[Hint: use the law of large numbers for Brownian motion.]

Solution

Since $\tau(t) \uparrow \infty$ almost surely as $t \uparrow \infty$, to study the behaviour of

$$\frac{\displaystyle\int_0^t \phi(u)\,dB(u)}{\displaystyle\int_0^t |\phi(u)|^2\,du} = \frac{X(t)}{A(t)}\ ,$$

as $t \uparrow \infty$, it is sufficient to study the behaviour of

$$\frac{X(\tau(t))}{A(\tau(t))} = \frac{Z(t)}{t}\ ,$$

as $t \uparrow \infty$. Note that the process Z is a Brownian motion, and it follows from the law of large numbers for Brownian motion that

$$\frac{Z(t)}{t} \longrightarrow 0\ ,$$

almost surely as $t \uparrow \infty$.

□

Sequential maximum likelihood estimation

Consider the following statistical model: there exist a parametric family $(\mathbb{P}_\theta\,,\ \theta \in \mathbb{R})$ of probability measures and a one–dimensional stochastic process X, such that under $\mathbb{P}_\theta$ it holds

$$dX(t) = \theta\, b(X(t))\,dt + dW_\theta(t)\ ,$$

where W_θ is a standard Brownian motion, and where the drift function b satisfies the *global Lipschitz* and *linear growth* conditions.

It is assumed that the maximum likelihood estimator of the parameter θ based on the observation of $(X(t)\,,\, 0 \leq t \leq T)$ in the time interval $[0, T]$ is given by the following expression

$$\widehat{\theta}(T) = \frac{\displaystyle\int_0^T b(X(t))\,dX(t)}{\displaystyle\int_0^T |b(X(t))|^2\,dt}\ .$$

Let θ_0 denote the (unknown) true value of the parameter, and let $\mathbb{P}_0 = \mathbb{P}_{\theta_0}$ denote the corresponding probability measure.

(viii) **Show that under $\mathbb{P}_0$ the maximum likelihood estimator satisfies**

$$\widehat{\theta}(T) = \theta_0 + \frac{\displaystyle\int_0^T b(X(t))\, dW_0(t)}{\displaystyle\int_0^T |b(X(t))|^2\, dt} \ ,$$

where W_0 is a standard Brownian motion.

Solution

Under $\mathbb{P}_0 = \mathbb{P}_{\theta_0}$ it holds

$$dX(t) = \theta_0\, b(X(t))\, dt + dW_0(t) \ ,$$

where $W_0 = W_{\theta_0}$ is a standard Brownian motion, hence

$$\begin{aligned}\int_0^T b(X(t))\, dX(t) &= \int_0^T b(X(t))\, [\theta_0\, b(X(t))\, dt + dW_0(t)] \\ &= \theta_0 \int_0^T |b(X(t))|^2\, dt + \int_0^T b(X(t))\, dW_0(t) \ ,\end{aligned}$$

and

$$\widehat{\theta}(T) = \frac{\displaystyle\int_0^T b(X(t))\, dX(t)}{\displaystyle\int_0^T |b(X(t))|^2\, dt} = \theta_0 + \frac{\displaystyle\int_0^T b(X(t))\, dW_0(t)}{\displaystyle\int_0^T |b(X(t))|^2\, dt} \ .$$

□

Note that this expression cannot be used in practice, since neither is $(W_0(t)\,,\ 0 \leq t \leq T)$ observed (available), nor is θ_0 known. The purpose of this expression is rather to analyze the behaviour of the estimator $\widehat{\theta}(T)$, for instance its asymptotic behaviour as $T \uparrow \infty$.

(ix) **Show that under $\mathbb{P}_0$ the maximum likelihood estimator is strongly consistent, i.e. $\widehat{\theta}(T) \to \theta_0$ almost surely as $T \uparrow \infty$.**

Solution

Using the result obtained in question (vii) yields

$$\frac{\displaystyle\int_0^T b(X(t))\, dW_0(t)}{\displaystyle\int_0^T |b(X(t))|^2\, dt} \longrightarrow 0 \ ,$$

almost surely as $T \uparrow 0$, hence under $\mathbb{P}_0$

$$\widehat{\theta}(T) = \theta_0 + \frac{\displaystyle\int_0^T b(X(t))\, dW_0(t)}{\displaystyle\int_0^T |b(X(t))|^2\, dt} \longrightarrow \theta_0 \ ,$$

almost surely as $T \uparrow 0$, i.e. the maximum likelihood estimator $\widehat{\theta}(T)$ is strongly consistent.

□

Actually, studying the ratio of two random variables is not so easy, and it is more convenient to study the time–changed estimator

$$\overline{\theta}(H) = \widehat{\theta}(\tau(H)) \qquad \text{where} \qquad \tau(H) = \inf\{T \geq 0 : \int_0^T |b(X(t))|^2\, dt = H\} \ .$$

(x) **Show that under $\mathbb{P}_0$ the time–changed maximum likelihood estimator satisfies**

$$\overline{\theta}(H) = \theta_0 + \frac{1}{H} \int_0^{\tau(H)} b(X(t))\, dW_0(t) \ .$$

Solution

Clearly

$$\int_0^{\tau(H)} |b(X(t))|^2\, dt = H \ ,$$

and under $\mathbb{P}_0$ it holds

$$\overline{\theta}(H) = \widehat{\theta}(\tau(H)) = \theta_0 + \frac{\displaystyle\int_0^{\tau(H)} b(X(t))\, dW_0(t)}{\displaystyle\int_0^{\tau(H)} |b(X(t))|^2\, dt} = \theta_0 + \frac{1}{H} \int_0^{\tau(H)} b(X(t))\, dW_0(t) \ .$$

□

The benefit of considering the time–changed maximum likelihood estimator is that the denominator is now deterministic, and the problem reduces to studying a stochastic integral under its intrinsic clock.

(xi) **Using the results obtained in the first part, show that under $\mathbb{P}_0$ the time–changed maximum likelihood estimator**

- **is strongly consistent, i.e. $\overline{\theta}(H) \to \theta_0$ almost surely as $H \uparrow \infty$,**
- **is unbiased (i.e. has a mean equal to the true value θ_0),**
- **has a (nonasymptotic) variance equal to $1/H$,**
- **is normally distributed, with mean θ_0 and variance $1/H$.**

Solution

It follows from the results obtained in the first part that the process Z defined by

$$Z(H) = \int_0^{\tau(H)} b(X(t))\, dW_0(t) \ ,$$

for any $H \geq 0$, is a Brownian motion under $\mathbb{P}_0$, and in particular $Z(H)$ is a Gaussian random variable with mean zero and variance H. Note that

$$\overline{\theta}(H) = \theta_0 + \frac{Z(H)}{H} \ ,$$

hence $(\overline{\theta}(H) - \theta_0)$ is a Gaussian random variable with mean zero and variance $1/H$, and in particular

$$\mathbb{E}_0[\overline{\theta}(H)] = \theta_0 \qquad \text{and} \qquad \mathbb{E}_0|\overline{\theta}(H) - \theta_0|^2 = \frac{1}{H} \ .$$

□