INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems II
Introduction to SDE's

Written Exam (aka DS)

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The objective is to study the one–dimensional (Lévy stochastic area) process

$$A(t) = \int_0^t B_1(s)\, dB_2(s) - \int_0^t B_2(s)\, dB_1(s) \ ,$$

where $B(t) = (B_1(t), B_2(t))$ is a two–dimensional standard Brownian motion, with $B(0) = 0$. The Lévy stochastic area is used for instance in the design of high–order numerical schemes for SDE's.

(i) **Show that $A(t)$ is an Itô process (and give its decomposition in terms of a usual integral and a stochastic integral).**

Solution

Clearly

$$A(t) = \int_0^t \begin{pmatrix} -B_2(s) & B_1(s) \end{pmatrix} \begin{pmatrix} dB_1(s) \\ dB_2(s) \end{pmatrix}$$

or in other words

$$A(t) = \int_0^t \psi(s)\, ds + \int_0^t \phi(s)\, dB(s) \ ,$$

with

$$\psi(s) = 0 \qquad \text{and} \quad \phi(s) = \begin{pmatrix} -B_2(s) & B_1(s) \end{pmatrix} \ .$$

□

(ii) **Show that $\mathbb{E}[A(t)] = 0$ and $\mathbb{E}[|A(t)|^2] = t^2$.**

Solution

Note that

$$\phi(s)\, \phi^*(s) = B_1^2(s) + B_2^2(s) \ ,$$

with the notations introduced in the answer to question (i), hence

$$\mathbb{E} \int_0^t \phi(s)\, \phi^*(s)\, ds = \int_0^t \mathbb{E}[B_1^2(s)]\, ds + \int_0^t \mathbb{E}[B_2^2(s)]\, ds = 2 \int_0^t s\, ds = t^2 < \infty \ ,$$

i.e. the integrand $\phi(s)$ belongs to $M^2([0,T])$ for any $T>0$. Therefore, the stochastic integral $A(t)$ is a square–integrable martingale, and it follows that

$$\mathbb{E}[A(t)] = 0 \qquad \text{and} \qquad \mathbb{E}[|A(t)|^2] = \mathbb{E} \int_0^t \phi(s)\, \phi^*(s)\, ds = t^2 \ .$$

□

To go further, i.e. beyond the expression of the first two moments, and to study the probability distribution of the r.v. $A(t)$, it is convenient to introduce the one–dimensional processes

$$C(t) = B_1^2(t) + B_2^2(t) = |B(t)|^2 \qquad \text{and} \qquad D(t) = B_1(t)\, B_2(t) \ .$$

In particular, it will be proved that the charactristic function satisfies

$$\mathbb{E}[\exp\{i\, u\, A(t)\}] = \frac{1}{\cosh(u\, t)} \ , \tag{$\star$}$$

for any real number u. Here, 'cosh' denotes the hyperbolic cosine function.

(iii) **Show that $C(t)$ and $D(t)$ are two Itô processes (and give their decompositions in terms of a usual integral and a stochastic integral).**

Solution

Introduce the function $f(x_1, x_2) = x_1^2 + x_2^2$, and note that

$$f'(x_1, x_2) = \begin{pmatrix} 2\, x_1 & 2\, x_2 \end{pmatrix} \qquad \text{and} \qquad f''(x_1, x_2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \ .$$

In addition $f(B(0)) = B_1^2(0) + B_2^2(0) = 0$. The Itô formula for the two–dimensional Brownian motion $B(t) = (B_1(t), B_2(t))$ and for the function $f(x_1, x_2) = x_1^2 + x_2^2$, yields

$$\begin{aligned} f(B(t)) \ &= \ f(B(0)) + \int_0^t f'(B(s))\, dB(s) + \tfrac{1}{2} \int_0^t \Delta f(B(s))\, ds \\ &= \ 2 \int_0^t \begin{pmatrix} B_1(s) & B_2(s) \end{pmatrix} \begin{pmatrix} dB_1(s) \\ dB_2(s) \end{pmatrix} + 2\, t \ , \end{aligned}$$

or in other words

$$C(t) = \int_0^t \psi(s)\, ds + \int_0^t \phi(s)\, dB(s)$$

with

$$\psi(s) = 2 \qquad \text{and} \qquad \phi(s) = 2 \begin{pmatrix} B_1(s) & B_2(s) \end{pmatrix} \ .$$

Equivalently

$$C(t) = |B(t)|^2 = 2 \int_0^t B_1(s)\, dB_1(s) + 2 \int_0^t B_2(s)\, dB_2(s) + 2\, t \ .$$

Similarly, introduce the function $f(x_1, x_2) = x_1\, x_2$, and note that

$$f'(x_1, x_2) = \begin{pmatrix} x_2 & x_1 \end{pmatrix} \qquad \text{and} \qquad f''(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

In addition $f(B(0)) = B_1(0)\, B_2(0) = 0$. The Itô formula for the two–dimensional Brownian motion $B(t) = (B_1(t), B_2(t))$ and for the function $f(x_1, x_2) = x_1\, x_2$, yields

$$\begin{aligned} f(B(t)) &= f(B(0)) + \int_0^t f'(B(s))\, dB(s) + \tfrac{1}{2} \int_0^t \Delta f(B(s))\, ds \\ &= \int_0^t \begin{pmatrix} B_2(s) & B_1(s) \end{pmatrix} \begin{pmatrix} dB_1(s) \\ dB_2(s) \end{pmatrix} , \end{aligned}$$

or in other words

$$D(t) = \int_0^t \psi(s)\, ds + \int_0^t \phi(s)\, dB(s)$$

with

$$\psi(s) = 0 \qquad \text{and} \quad \phi(s) = \begin{pmatrix} B_2(s) & B_1(s) \end{pmatrix} .$$

Equivalently

$$D(t) = \int_0^t B_2(s)\, dB_1(s) + \int_0^t B_1(s)\, dB_2(s) .$$

□

(iv) **Check that $A(t)$ and $-A(t)$ have the same probability distribution, hence the probability distribution is symmetric. Show that**

$$\mathbb{E}[\exp\{i\, u\, A(t)\}] = \mathbb{E}[\cos(u\, A(t))] ,$$

for any real number u.

Solution

Clearly, the two processes $(B_1(t), B_2(t))$ and $(B_2(t), B_1(t))$ have the same probability distribution, hence the two r.v.'s

$$\int_0^t B_1(s)\, dB_2(s) - \int_0^t B_2(s)\, dB_1(s) \qquad \text{and} \qquad \int_0^t B_2(s)\, dB_1(s) - \int_0^t B_1(s)\, dB_2(s) ,$$

have the same probability distribution, or in other words $A(t)$ and $-A(t)$ have the same probability distribution. Therefore

$$\begin{aligned} &\mathbb{E}[\exp\{i\, u\, A(t)\}] = \mathbb{E}[\cos(u\, A(t))] + i\, \mathbb{E}[\sin(u\, A(t))] \\ &= \mathbb{E}[\exp\{-i\, u\, A(t)\}] = \mathbb{E}[\cos(u\, A(t))] - i\, \mathbb{E}[\sin(u\, A(t))] \\ &= \mathbb{E}[\cos(u\, A(t))] , \end{aligned}$$

for any real number u.

□

Define also the one–dimensional processes

$$\begin{aligned} V(t) &= \cos(u\,A(t)) \;, \\ W(t) &= -\tfrac{1}{2}\,\alpha(t)\,C(t) + \beta(t) \;, \\ Z(t) &= V(t)\,\exp\{W(t)\} \;, \end{aligned}$$

where $\alpha(t)$ and $\beta(t)$ are two continuously differentiable functions defined on $[0,\infty)$ with values in $\mathbb{R}$, to be specified later on.

(v) **Show that $V(t)$ and $W(t)$ are two Itô processes (and give their decompositions in terms of a usual integral and a stochastic integral).**

Solution

Recall from the answer to question (i) that $A(t)$ is a one–dimensional Itô process defined as

$$A(t) = \int_0^t \psi(s)\,ds + \int_0^t \phi(s)\,dB(s) \;,$$

with

$$\psi(s) = 0 \qquad \text{and} \quad \phi(s) = \begin{pmatrix} -B_2(s) & B_1(s) \end{pmatrix} \;.$$

Introduce the function $f(a) = \cos(u\,a)$ defined on $\mathbb{R}$ with values in $\mathbb{R}$, and note that

$$f'(a) = -u\,\sin(u\,a) \qquad \text{and} \qquad f''(a) = -u^2\,\cos(u\,a) = -u^2\,f(a) \;.$$

In addition $f(A(0)) = \cos(u\,A(0)) = 1$. The Itô formula for the one–dimensional Itô process $A(t)$ and for the function $f(a) = \cos(u\,a)$, yields

$$\begin{aligned} f(A(t)) &= f(A(0)) + \int_0^t f'(A(s))\,[\psi(s)\,ds + \phi(s)\,dB(s)] + \tfrac{1}{2}\int_0^t f''(A(s))\,\phi(s)\,\phi^*(s)\,ds \\ &= 1 - u\int_0^t \sin(u\,A(s))\,[B_1(s)\,dB_2(s) - B_2(s)\,dB_1(s)] \\ &\quad - \tfrac{1}{2}\,u^2\int_0^t f(A(s))\,[B_1^2(s) + B_2^2(s)]\,ds \end{aligned}$$

in other words

$$V(t) = 1 - \tfrac{1}{2}\,u^2\int_0^t V(s)\,C(s)\,ds - u\int_0^t \sin(u\,A(s))\,[B_1(s)\,dB_2(s) - B_2(s)\,dB_1(s)] \;.$$

Note that $X(t) = (\alpha(t), C(t))$ is a two–dimensional Itô process defined as

$$X(t) = X(0) + \int_0^t \begin{pmatrix} \dot\alpha(s) \\ \psi(s) \end{pmatrix} ds + \int_0^t \begin{pmatrix} 0 \\ \phi(s) \end{pmatrix} dB(s) \;,$$

with

$$\psi(s) = 2 \qquad \text{and} \qquad \phi(s) = 2\,\begin{pmatrix} B_1(s) & B_2(s) \end{pmatrix}\ .$$

Introduce the function $f(a,c) = a\,c$, and note that

$$f'(a,c) = \begin{pmatrix} c & a \end{pmatrix} \qquad \text{and} \qquad f''(a,c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ .$$

In addition $f(X(0)) = \alpha(0)\,C(0) = 0$. The Itô formula for the two–dimensional Itô process $X(t) = (\alpha(t), C(t))$ and for the function $f(a,c) = a\,c$, yields

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))\,[\begin{pmatrix} \dot\alpha(s) \\ \psi(s) \end{pmatrix} ds + \begin{pmatrix} 0 \\ \phi(s) \end{pmatrix} dB(s)\,] \\ &\quad + \tfrac{1}{2} \int_0^t \text{trace}(f''(X(s))\,\begin{pmatrix} 0 \\ \phi(s) \end{pmatrix} \begin{pmatrix} 0 & \phi^*(s) \end{pmatrix})\,ds \\ &= \int_0^t \begin{pmatrix} C(s) & \alpha(s) \end{pmatrix}\,[\begin{pmatrix} \dot\alpha(s) \\ \psi(s) \end{pmatrix} ds + \begin{pmatrix} 0 \\ \phi(s) \end{pmatrix} dB(s)\,] \\ &\quad + \tfrac{1}{2} \int_0^t \text{trace} \begin{pmatrix} 0 & \phi(s)\,\phi^*(s) \\ 0 & 0 \end{pmatrix} ds\ , \end{aligned}$$

in other words

$$\begin{aligned} \alpha(t)\,C(t) &= \int_0^t [C(s)\,\dot\alpha(s) + 2\,\alpha(s)]\,ds \\ &\quad + 2 \int_0^t \alpha(s)\,[B_1(s)\,dB_1(s) + B_2(s)\,dB_2(s)]\ . \end{aligned}$$

Finally

$$\begin{aligned} W(t) &= -\tfrac{1}{2}\,\alpha(t)\,C(t) + \beta(t) \\ &= \beta(0) + \int_0^t [\dot\beta(s) - \tfrac{1}{2}\,C(s)\,\dot\alpha(s) - \alpha(s)]\,ds \\ &\quad - \int_0^t \alpha(s)\,[B_1(s)\,dB_1(s) + B_2(s)\,dB_2(s)]\ . \end{aligned}$$

□

(vi) **Finally, show that $Z(t)$ is an Itô process (and give its decomposition in terms of a usual integral and a stochastic integral).**

Give ODE's that the functions $\alpha(t)$ and $\beta(t)$ should satisfy, for the usual integral vanish in the decomposition.

Solution

Note that $X(t) = (V(t), W(t))$ is a two–dimensional Itô process defined as

$$X(t) \;=\; X(0) + \int_0^t \psi(s)\, ds + \int_0^t \phi(s)\, dB(s) \;,$$

with

$$\psi(s) = \begin{pmatrix} -\frac{1}{2}\, u^2\, V(s)\, C(s) \\ \dot{\beta}(s) - \frac{1}{2}\, C(s)\, \dot{\alpha}(s) - \alpha(s) \end{pmatrix} \;,$$

and

$$\phi(s) = \begin{pmatrix} u\, \sin(u\, A(s))\, B_2(s) & -u\, \sin(u\, A(s))\, B_1(s) \\ -\alpha(s)\, B_1(s) & -\alpha(s)\, B_2(s) \end{pmatrix} \;.$$

Note that

$$\phi(s)\, \phi^*(s) = \begin{pmatrix} u^2\, \sin^2(u\, A(s)) & 0 \\ 0 & \alpha^2(s) \end{pmatrix} \, C(s)$$

Introduce the function $f(v, w) = v\, \exp\{w\}$, and note that

$$f'(v, w) = \begin{pmatrix} \exp\{w\} & v\, \exp\{w\} \end{pmatrix} \qquad \text{and} \qquad f''(v, w) = \begin{pmatrix} 0 & \exp\{w\} \\ \exp\{w\} & v\, \exp\{w\} \end{pmatrix} \;.$$

Note that

$$\begin{aligned} f''(X(s))\, \phi(s)\, \phi^*(s) \;&=\; \begin{pmatrix} 0 & \exp\{W(s)\} \\ \exp\{W(s)\} & Z(s) \end{pmatrix} \begin{pmatrix} u^2\, \sin^2(u\, A(s)) & 0 \\ 0 & \alpha^2(s) \end{pmatrix} C(s) \\ &= \begin{pmatrix} 0 & \alpha^2(s)\, \exp\{W(s)\} \\ u^2\, \sin^2(u\, A(s))\, \exp\{W(s)\} & \alpha^2(s)\, Z(s) \end{pmatrix} C(s) \;, \end{aligned}$$

and

$$\mathrm{trace}(f''(X(s))\, \phi(s)\, \phi^*(s)) = \alpha^2(s)\, Z(s)\, C(s) \;.$$

In addition $f(X(0)) = V(0)\, \exp\{W(0)\} = \exp\{\beta(0)\}$.

The Itô formula for the two–dimensional Itô process $X(t) = (V(t), W(t))$ and for the function $f(v,w) = v \exp\{w\}$, yields

$$\begin{aligned}
f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))\, [\psi(s)\, ds + \phi(s)\, dB(s)] \\
&\quad + \tfrac{1}{2} \int_0^t \mathrm{trace}(f''(X(s))\, \phi(s)\, \phi^*(s))\, ds \\
&= \exp\{\beta(0)\} + \int_0^t \begin{pmatrix} \exp\{W(s)\} & V(s)\, \exp\{W(s)\} \end{pmatrix} \\
&\qquad [\begin{pmatrix} -\tfrac{1}{2}\, u^2\, V(s)\, C(s) \\ \dot\beta(s) - \tfrac{1}{2}\, C(s)\, \dot\alpha(s) - \alpha(s) \end{pmatrix} ds \\
&\qquad + \begin{pmatrix} u\, \sin(u\, A(s))\, B_2(s) & -u\, \sin(u\, A(s))\, B_1(s) \\ -\alpha(s)\, B_1(s) & -\alpha(s)\, B_2(s) \end{pmatrix} \begin{pmatrix} dB_1(s) \\ dB_2(s) \end{pmatrix}] \\
&\quad + \tfrac{1}{2} \int_0^t \alpha^2(s)\, Z(s)\, C(s)\, ds \ ,
\end{aligned}$$

in other words

$$\begin{aligned}
Z(t) &= \exp\{\beta(0)\} \\
&\quad + \int_0^t [-\tfrac{1}{2}\, u^2\, Z(s)\, C(s) + (\dot\beta(s) - \tfrac{1}{2}\, C(s)\, \dot\alpha(s) - \alpha(s))\, Z(s) + \tfrac{1}{2}\, \alpha^2(s)\, Z(s)\, C(s)]\, ds \\
&\quad + \int_0^t \begin{pmatrix} \exp\{W(s)\} & Z(s) \end{pmatrix} \begin{pmatrix} u\, \sin(u\, A(s))\, [B_2(s)\, dB_1(s) - B_1(s)\, dB_2(s)] \\ -\alpha(s)\, [B_1(s)\, dB_1(s) + B_2(s)\, dB_2(s)] \end{pmatrix} .
\end{aligned}$$

A sufficient condition for the usual integral to vanish is

$$-\tfrac{1}{2}\, u^2\, Z(s)\, C(s) + (\dot\beta(s) - \tfrac{1}{2}\, C(s)\, \dot\alpha(s) - \alpha(s))\, Z(s) + \tfrac{1}{2}\, \alpha^2(s)\, Z(s)\, C(s) = 0 \ ,$$

or

$$(\dot\beta(s) - \alpha(s))\, Z(s) + \tfrac{1}{2}\, (-u^2 - \dot\alpha(s) + \alpha^2(s))\, Z(s)\, C(s) = 0 \ ,$$

which is satisfied, provided that the functions $\alpha(t)$ and $\beta(t)$ satisfy the following system of ODE's

$$\begin{aligned}
\dot\beta(s) &= \alpha(s) \ , \\
\dot\alpha(s) &= \alpha^2(s) - u^2 \ .
\end{aligned}$$

□

(vii) **Let $T > 0$ be fixed. Check that the two functions**

$$\alpha(t) = u\, \tanh((T-t)\, u) \qquad \textbf{and} \qquad \beta(t) = -\log\cosh((T-t)\, u) \ ,$$

satisfy the ODE's introduced in the answer to question (vi).

Here, 'tanh' denotes the hyperbolic tangent function.

Solution

Clearly

$$\dot{\alpha}(t) = -u^2\,[1 - \tanh^2((T-t)\,u)] = -u^2 + \alpha^2(t)\ ,$$

and

$$\dot{\beta}(t) = -\ \frac{-u\,\sinh((T-t)\,u)}{\cosh((T-t)\,u)} = u\,\tanh((T-t)\,u) = \alpha(t)\ .$$

□

(viii) **Let $\alpha(t)$ and $\beta(t)$ be defined as in question (vii). Check that the process $Z(t)$ is a square–integrable martingale in this case.**

Solution

With this choice of the functions $\alpha(t)$ and $\beta(t)$, it holds

$$\begin{aligned} Z(t) \ &= \ \exp\{\beta(0)\} \\ &+ \int_0^t \big(\,\exp\{W(s)\} \quad Z(s)\,\big) \begin{pmatrix} u\,\sin(u\,A(s))\,[B_2(s)\,dB_1(s) - B_1(s)\,dB_2(s)] \\ -\alpha(s)\,[B_1(s)\,dB_1(s) + B_2(s)\,dB_2(s)] \end{pmatrix} . \end{aligned}$$

For this expression to define a square–integrable martingale, the four integrands

$$\exp\{W(s)\}\,\sin(u\,A(s))\,B_2(s) \qquad \text{and} \qquad \alpha(s)\,Z(s)\,B_1(s)\ ,$$

and

$$\exp\{W(s)\}\,\sin(u\,A(s))\,B_1(s) \qquad \text{and} \qquad \alpha(s)\,Z(s)\,B_2(s)\ ,$$

should belong to $M^2([0,T])$, and it is enough to check that the first two integrands

$$\exp\{W(s)\}\,\sin(u\,A(s))\,B_2(s) \qquad \text{and} \qquad \alpha(s)\,\exp\{W(s)\}\,\cos(u\,A(s))\,B_1(s)\ ,$$

belong to $M^2([0,T])$. Note that $\alpha(s) \geq 0$, $\beta(s) \leq 0$ and $C(s) \geq 0$ for any $0 \leq s \leq T$, hence $W(s) \leq 0$ and $0 \leq \exp\{W(s)\} \leq 1$ for any $0 \leq s \leq T$. Note also that $0 \leq \alpha(s) \leq \alpha(0)$ for any $0 \leq s \leq T$. Therefore

$$|\,\exp\{W(s)\}\,\sin(u\,A(s))\,B_2(s)| \leq |B_2(s)|\ ,$$

and

$$|\alpha(s)\,\exp\{W(s)\}\,\cos(u\,A(s))\,B_1(s)| \leq \alpha(0)\,|B_1(s)|\ .$$

□

(ix) **Conclude and check that**

$$\mathbb{E}[\exp\{i\,u\,A(t)\}] = \frac{1}{\cosh(u\,t)}\ . \qquad (\star)$$

holds for any real number u.

Solution

It holds
$$\mathbb{E}[Z(T)] = \mathbb{E}[Z(0)] \ ,$$
since $Z(t)$ is a square–integrable martingale, and note that
$$Z(T) = V(T)\ \exp\{-\tfrac{1}{2}\,\alpha(T)\,C(T) + \beta(T)\} = \cos(u\,A(T)) \ ,$$
and
$$Z(0) = V(0)\ \exp\{-\tfrac{1}{2}\,\alpha(0)\,C(0) + \beta(0)\} = \exp\{\beta(0)\} = \frac{1}{\cosh(u\,T)} \ ,$$
hence
$$\mathbb{E}[\exp\{i\,u\,A(T)\}] = \mathbb{E}[\cos(u\,A(T))] = \mathbb{E}[Z(T)] = \mathbb{E}[Z(0)] = \frac{1}{\cosh(u\,T)} \ ,$$
for any real number u.

□