Bearings-Only Tracking in Modified Polar Coordinate System : Initialization of the Particle Filter and Posterior Cramér-Rao Bound

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Summary

1. What is the Bearings-Only Tracking problem ?

2. Solving using particle filtering algorithm...

3. From cartesian to modified polar coordinate system.

4. Initialization of the particle filtering algorithm.

5. Posterior Cramér-Rao bound.

6. Conclusion.

1. What is the Bearings-Only Tracking problem ? (1/2)

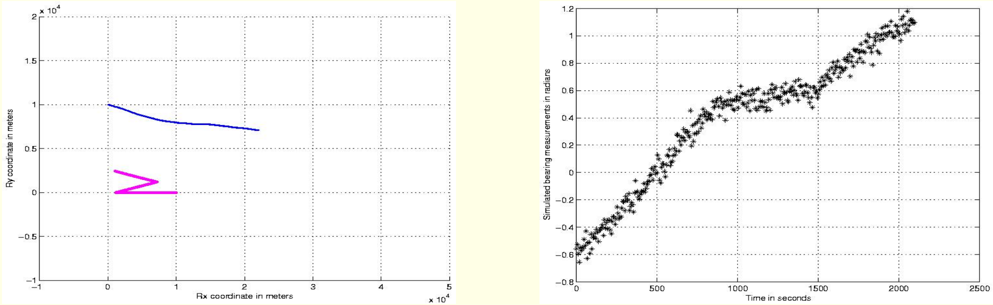


FIG. 1 – *Trajectories of the observer (pink) and the target (blue) and simulated bearing measurements.*

$$X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \\ r_x(t) \\ r_y(t) \end{pmatrix} \qquad Z_t \approx \tan\left(\frac{r_y(t)}{r_x(t)}\right)$$

- X_t is the target state at time t composed of relative velocity and position of the target in the $x - y$ plane
- Z_t is the bearing measurement received at time t.

Problem : Estimate X_t using $\{Z_1, \ldots, Z_t\}$.

1. What is the bearings-only tracking problem ?(2/2)

The stochastic system :

$$\begin{cases} X_{t+1} = AX_t + HU_t + W_t \\ Z_t = G(X_t) + V_t \end{cases}$$

where :

- X_t is the target state at time t composed of relative velocity and position of the target in the $x - y$ plane,
- Z_t the bearing measurement received at time t,
- U_t is the known difference between observer velocity at time $t + 1$ and t.
- V_t has a center normal distribution with variance σ_v known.
- W_t has a center normal distribution with covariance matrix Q known.

This is a non-linear filtering problem which can be solved using particle filtering algorithm !

2. Solving using particle filtering algorithm...

At each step of time :

1. Propagating the set of particles using the state equation.
2. Weighting each of the particles using the measurement equation.
3. Resampling step.

Reference : Doucet et al. (2001)

Problem

Particles must be properly initialized !

3. From cartesian to modified polar coordinate system (1/3)

There is an unobservability problem hidden in the cartesian formulation of the system :

$$\begin{cases} X_{t+1} = AX_t + HU_t + W_t \\ Z_t = G(X_t) + V_t \end{cases}$$

Problem

The range is unobservable until the observer has maneuvered.

Solution

A coordinate system more suited to the problem :
the modified polar coordinate system

Reference : Aidala and Hammel (1983)

3. From cartesian to modified polar coordinate system (2/3)

The modified polar coordinates

$$Y(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \\ Y_4(t) \end{pmatrix} = \begin{pmatrix} \dot{\beta}(t) \\ \frac{\dot{r}(t)}{r(t)} \\ \beta(t) \\ \frac{1}{r(t)} \end{pmatrix}$$

We can show that :

- $Y_4(t)$ is unobservable until the observer has not maneuvered.
- $Y^r(t) = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{pmatrix} = \begin{pmatrix} \dot{\beta}(t) \\ \frac{\dot{r}(t)}{r(t)} \\ \beta(t) \end{pmatrix}$ is always observable.

3. From cartesian to modified polar coordinate system (3/3)

Before the observer maneuvers, the stochastic system in modified polar coordinate system is

$$\begin{cases} Y^r_{t+1} = F(Y^r_t, \tilde{W}_t), \\ Y_4(t+1) = H(Y_t, \tilde{W}_t), \\ Z_t = Y_3(t) + V_t \end{cases}$$

where

$$\tilde{W}_t = Y_4(t) W_t$$

and W_t has a center normal distribution with covariance matrix Q known.

An interesting model :
This is a non linear filtering problem with unknown covariance state
($Y_4(t)$ is unknown !).

4. Initialization of the particle filtering algorithm (1/7)

2 key ideas :

– We can proove that if the target has a deterministic trajectory then for all k

$$Z(t_0+k) = Y_3(t_0) + \tan^{-1}\left(\frac{k\delta_t Y_1(t_0)}{1+k\delta_t Y_2(t_0)}\right) + V_k$$

It is just an optimization problem.
The observable components $Y_{t_0}^r$ can be estimated using the set of measurements $\{Z_{t_0}, \ldots, Z_{t_0+K}\}$.

– We only assume a prior information on the unobservable component $Y_4(t_0)$:

$$\frac{1}{R_{max}} \leq Y_4(t_0) \leq \frac{1}{R_{min}}$$

4. Initialization of the particle filtering algorithm (2/7)

Initialization of the particle filtering algorithm

Wait until time $t_0 + K$, the particule i is initialized by

1. $$\begin{pmatrix} Y_1^{(i)}(t_0) \\ Y_2^{(i)}(t_0) \\ Y_3^{(i)}(t_0) \end{pmatrix} \sim CA(\hat{Y}_{t_0}^r),$$

   where

   – $\hat{Y}_{t_0}^r$ is computed using a Gauss-Newton optimization algorithm (initialized by linear regression).
   – $CA(\hat{Y}_{t_0}^r)$ is the confidence area of $\hat{Y}_{t_0}^r$ (approximated by an hyperellipsoid).

2. $$Y_4^{(i)}(t_0) \sim \frac{1}{\mathcal{U}[R_{min}, R_{max}]}.$$

4. Initialization of the particle filtering algorithm (3/7)

Two important points

– Before the observer maneuvers,

the observable component of the particles $\begin{pmatrix} Y_1^{(i)}(t) \\ Y_2^{(i)}(t) \\ Y_3^{(i)}(t) \end{pmatrix}$ and the unobservable component of the particles $Y_4^{(i)}(t)$ must be resampled independently !

– How the initialization time K can be fixed ?

”The particle filtering algorithm is initialized as soon as the volume of the confidence area for $\hat{Y}_{t_0}^r$ is sufficiently small to be filled by N particles”.

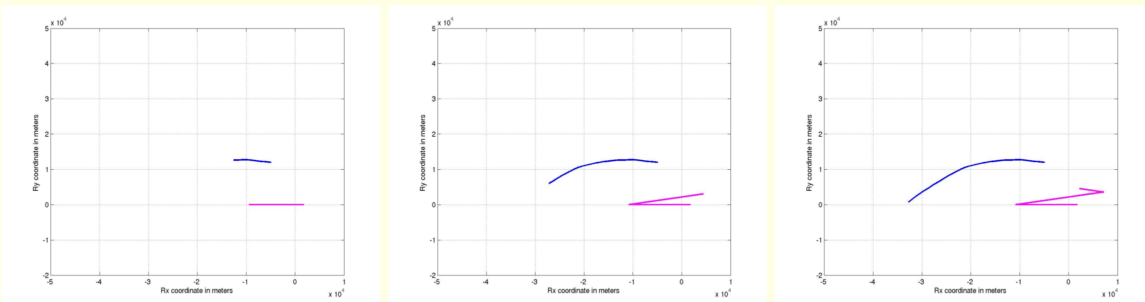
4. Initialization of the particle filtering algorithm (4/7)

Simulation scenario

$$X_0^{target} = \begin{pmatrix} 10 \text{ ms}^{-1} \\ -2 \text{ ms}^{-1} \\ 0 \text{ m} \\ 10000 \text{ m} \end{pmatrix}, X_0^{obs} = \begin{pmatrix} -6 \text{ ms}^{-1} \\ 3 \text{ ms}^{-1} \\ 10000 \text{ m} \\ 0 \text{ m} \end{pmatrix}.$$

The observer follows a leg-by-leg trajectory. His velocity vector is constant on each leg and modified at the two following instants :

$$\begin{pmatrix} V_x^{obs}(2100) \\ V_x^{obs}(2100) \end{pmatrix} = \begin{pmatrix} 10 \text{ ms}^{-1} \\ 2 \text{ ms}^{-1} \end{pmatrix}, \begin{pmatrix} V_x^{obs}(3900) \\ V_x^{obs}(3900) \end{pmatrix} = \begin{pmatrix} -10 \text{ ms}^{-1} \\ 2 \text{ ms}^{-1} \end{pmatrix}.$$

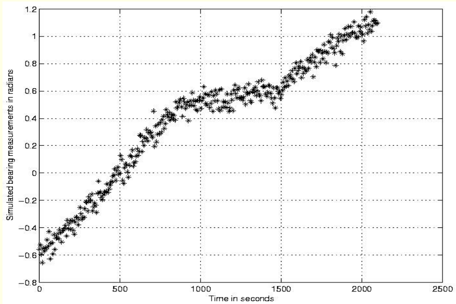


$t = 1800s$ $t = 3600s$ $t = 5400s$

Trajectories of the observer (pink) and the target (blue).

4. Initialization of the particle filtering algorithm (5/7)

Simulation scenario



Simulated bearing measurements.
The measurement standard deviation is 0.05 rad (about 3 deg).

Parameters for the particle filtering algorithm

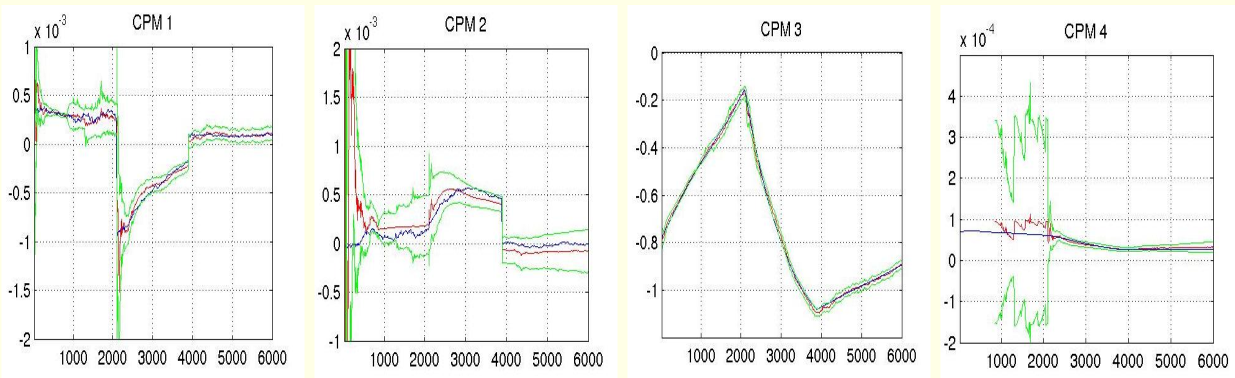
- Number of particles : 10000,
- Sampling threshold : 0.5,

The single assumption : $R_{min} = 1000$ m and $R_{max} = 40000$ m.

4. Initialization of the particle filtering algorithm (6/7)

Simulation results in modified polar coordinates

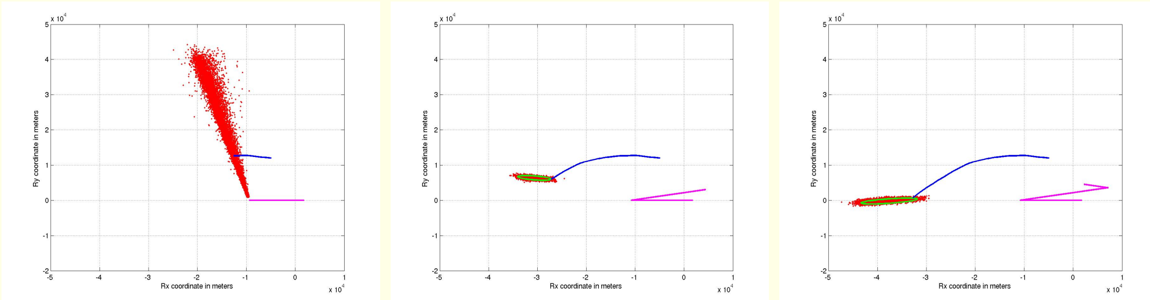
$Y_1(t)$ $Y_2(t)$ $Y_3(t)$ $Y_4(t)$



True value (blue), Estimates (red), Confidence (green)

4. Initialization of the particle filtering algorithm (7/7)

Simulation results in cartesian system



$t = 1800s$ $t = 3600s$ $t = 5400s$

Trajectory of the observer (pink), trajectory of the true target (blue), particles (red), confidence for the estimate (green).

Conclusion

– We have initialized the particle filtering algorithm using a weak prior on range.
– Performance analysis in polar modified coordinate system.

5. Posterior Cramér-Rao Bound (1/9)

Next step :

The performance analysis in modified polar coordinate system.

Tool :

The Posterior Cramér-Rao Bound (PCRB).

”The PCRB gives a lower bound for the Mean Square Error”

5. Posterior Cramér-Rao Bound (2/9)

Notations : $Y_{0:t} = \{Y_0, \dots, Y_t\}$ and $Z_{0:t} = \{Z_0, \dots, Z_t\}$.

The bias

$$B(Y_{0:t}) = E\left(g(Z_{0:t}|Y_{0:t})\right) - Y_{0:t}$$

where $g(Z_{0:t})$ is the estimator of $Y_{0:t}$

The asymptotic bias assumption

$$\lim_{Y_i(j)\to\mathcal{Y}_i^+} B(Y_{0:t})p(Y_{0:t}) = \lim_{Y_i(j)\to\mathcal{Y}_i^-} B(Y_{0:t})p(Y_{0:t})$$
$$\forall i \in \{1, \dots, n_y\} \text{ and } j \in \{0, \dots, t\}$$

where

– $\mathcal{Y}_i$ is the state space of $Y_i(j)$ for all j in $\{0, \dots, t\}$,

– $\mathcal{Y}_i^-$ and $\mathcal{Y}_i^+$ are the endpoints of $\mathcal{Y}_i$

5. Posterior Cramér-Rao Bound (3/9)

Proposition 1

Under the asymptotic bias assumption,

$$ECM_{0:t} \succcurlyeq J_{0:t}^{-1}$$

where

$$\begin{aligned} ECM_{0:t} &= E\{(Y_{0:t} - g(Z_{0:t}))(Y_{0:t} - g(Z_{0:t}))^t\} \\ J_{0:t} &= E\{-\Delta_{Y_{0:t}}^{Y_{0:t}} \ln p(Z_{0:t}, Y_{0:t})\} \end{aligned}$$

and $g(Z_{0:t})$ is the estimator of $Y_{0:t}$.

In the filtering context, we only want to approximate the right lower block of $J_{0:t}$ noted J_t.

5. Posterior Cramér-Rao Bound (4/9)

Using Tichavsky et al. results, we have

$$J_{t+1} = D_t^{22} - D_t^{21}(J_t + D_t^{11})^{-1} D_t^{12}$$

where

$$\begin{aligned}
D_t^{11} &= E\{\nabla_{Y_t} \ln p(Y_{t+1}|Y_t) \nabla_{Y_t}^t \ln p(Y_{t+1}|Y_t)\}\}, \\
D_t^{21} &= E\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t) \nabla_{Y_t}^t \ln p(Y_{t+1}|Y_t)\}, \\
D_t^{12} &= E\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t) \nabla_{Y_{t+1}}^t \ln p(Y_{t+1}|Y_t)\}, \\
D_t^{22} &= E\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t) \nabla_{Y_{t+1}}^t \ln p(Y_{t+1}|Y_t)\} \\
&+ E\{\nabla_{Y_{t+1}} \ln p(Z_{t+1}|Y_{t+1}) \nabla_{Y_{t+1}}^t \ln p(Z_{t+1}|Y_{t+1})\}.
\end{aligned}$$

Conclusion :

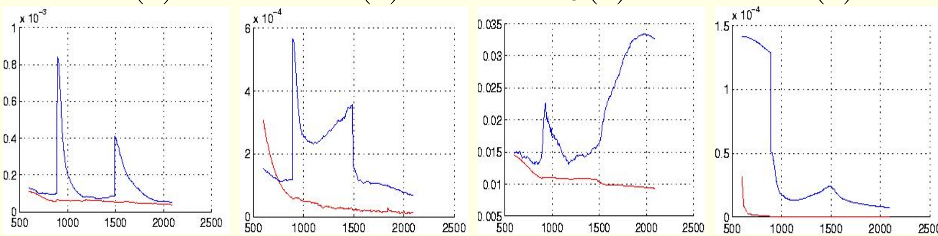
J_t can be computed recursively .

Rq : D_t^{11}, D_t^{12}, D_t^{22}, D_t^{21} are approximated using Monte Carlo method and $J(t_0)$ is obtained by a Cramér-Rao Bound.

5. Posterior Cramér-Rao Bound (5/9)

Simulation results

$Y_1(t)$ $Y_2(t)$ $Y_3(t)$ $Y_4(t)$



(Mean square error in blue, PCRB in red)

Problem :

The PCRB seems over optimistic...

Hypothesis :

– Is the asymptotic bias assumption true ?
– Is $J_{0:t}$ ill-conditionned due to range unobservability ?

5. Posterior Cramér-Rao Bound (6/9)

Is the asymptotic bias assumption true ?

A more general bound :

$$ECM_{0:t} \succcurlyeq C_{0:t} J_{0:t}^{-1} C_{0:t}^{t}$$

where

$$C_{0:t} = E\{(g(Z_{0:t}) - Y_{0:t}) \nabla_{Y_{0:t}}^{t} \ln p(Z_{0:t}, Y_{0:t})\}.$$

Remarks :

– $C_{0:t}$ can be approximated using Monte Carlo approximation.

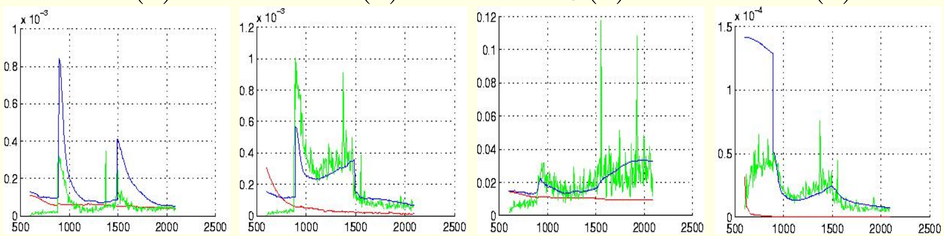
– The recursive formulation of the PCRB is no longer valid.

5. Posterior Cramér-Rao Bound (7/9)

Is the asymptotic bias assumption true ?

Simulation results

$Y_1(t)$ $Y_2(t)$ $Y_3(t)$ $Y_4(t)$



(Mean square error in blue, classical PCRB in red, new PCRB in green)

Solution :

The asymptotic bias assumption is not true in the bearings-only tracking problem.

5. Posterior Cramér-Rao Bound (8/9)

Problem :

The posterior Cramér-Rao bound seems over optimistic...

Hypothesis :

– Is the asymptotic bias assumption true ? No
– Is $J_{0:t}$ ill-conditionned due to range unobservability ?

5. Posterior Cramér-Rao Bound (9/9)

Is $J_{0:t}$ ill-conditionned due to range unobservability ?

Idea : We only construct a bound for the observable components of the state Y^r_t.

We can proove that :

$$ECM^r_{0:t} \succcurlyeq E\{C_{0:t}(Y_4(t_0))J^{-1}_{0:t}(Y_4(t_0))C^t_{0:t}(Y_4(t_0))\}.$$

where

$$\begin{aligned} J_{0:t}(Y_4(t_0)) &= E\{-\Delta^{Y^r_{0:t}}_{Y^r_{0:t}} \ln p_{Y_4(t_0)}(Z_{0:t}, Y^r_{0:t})|Y_4(t_0)\}, \\ C_{0:t}(Y_4(t_0)) &= E\{(g^r(Z_{0:t}) - Y^r_{0:t})\nabla^t_{Y^r_{0:t}} \ln p_{Y_4(t_0)}(Z_{0:t}, Y^r_{0:t})|Y_4(t_0)\}. \end{aligned}$$

It is possible to approximate $J_{0:t}(Y_4(t_0))$ and $C_{0:t}(Y_4(t_0))$.

6. Conclusion

We have proposed :

- An initialization method for particle filtering algorithm using modifed polar coordinates using a weak prior on range.
- A new interpretation of the Bearings-Only Tracking problem.
- A realistic posterior Cramér-Rao bound in modified polar coordinate system.

Perspectives :

- 3D target tracking.
- Maneuvering target.