The theory of successor extended by several predicates

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Abstract

In [ER66], Elgot and Rabin devise a method for constructing unary predicates P such that the MSO theory of $\langle \mathbb{N}, +1, P\rangle$ is decidable (here $+1$ denotes the successor relation). Further results in this direction have been established in ([Sie70],[Sem84],[Mae99],[CT02],[FS03]).

This kind of problem takes place in the more general perspective of studying "weak" arithmetical theories, which possess interesting decidability properties ([Bès01]).

We present here a method allowing to define infinite sequences of monadic predicates $P_1, \dots, P_n$, such that the MSO theory of $\langle \mathbb{N}, +1, (P_i)_{i\in\mathbb{N}}\rangle$ is decidable.

In particular, we build such predicate P_i that can have very slow "growth"; i.e., the function associating to k the k-th element of P_i can be comparable to $\lfloor nlogn \rfloor$, $\lfloor nlog(logn)) \rfloor$ or even $\lfloor nlog^*(n) \rfloor$.

As in [FS03], the method consists of consider integer sequences computed by k-automata. The new feature of the automata here considered is that transitions are "controlled" by some predicates.

1 Preliminaries

1.1 Extended Iterated Pushdown Automata

1.1.1 Iterated pushdown stores

Originally defined by [Gre70], Iterated-pushdown stores are storage structures built iteratively. Here, we shall use the definition of [DG86] and stick to their notation.

Definition 1 (k-iterated pushdown store). *Let Γ be a set. We define inductively the set k-$\mathsf{pds}(\Gamma)$ of k-iterated pushdown-stores over Γ:*

$$0\text{-}\mathsf{pds}(\Gamma) = \{\varepsilon\}, (k+1)\text{-}\mathsf{pds}(\Gamma) = (\Gamma[k\text{-}\mathsf{pds}(\Gamma)])^*, it\text{-}\mathsf{pds}(\Gamma) = \bigcup_{k\geq 0} k\text{-}\mathsf{pds}(\Gamma).$$

From the definition, every non empty ω in $(k+1)$-$\mathsf{pds}(\Gamma)$, $k \geq 1$, has a unique decomposition as

$$\omega = a[\omega_1]\omega'$$

with $\omega_1 \in k$-$\mathsf{pds}(\Gamma)$, $\omega' \in (k+1)$-$\mathsf{pds}(\Gamma) \cup \{\varepsilon\}$ and $a \in \Gamma$. In the rest of the paper, we will often replace by a every occurence of $a[\varepsilon]$ appearing in the description of a k-pds.

Example 2. *Let $\Gamma = \{a_1, b_1, a_2, b_2, a_3, b_3\}$ be a storage alphabet, we consider the following 3-pds:* $\omega_{ex} = b_3[b_2[b_1[\varepsilon]a_1[\varepsilon]]a_2[a_1[\varepsilon]]]a_3[\varepsilon]a_3[a_2[a_1[\varepsilon]b_1[\varepsilon]]] \in 3\text{-}\mathsf{pds}(\Gamma)$.
ω_{ex} will be writen

$$\omega_{ex} = b_3[b_2[b_1a_1]a_2[a_1]]a_3a_3[a_2[a_1b_1]],$$

and its decomposition corresponds to $a = b_3$, $\omega_1 = b_2[b_1a_1]a_2[a_1]$ and $\omega' = a_3a_3[a_2[a_1b_1]]$.

We now formalize operations allowed on the store.

Definition 3 (The reading operation). *The map* top : it-$\mathsf{pds}(\Gamma) \to \Gamma^*$ *is defined by*

$$\mathrm{top}(\varepsilon) = \varepsilon, \ \mathrm{top}(a[\omega_1]\omega) = a \cdot \mathrm{top}(\omega_1).$$

Definition 4 (The pop **operation at level** j**).** *The map* $\mathrm{pop}_j : it\text{-}\mathsf{pds}(\Gamma) \to it\text{-}\mathsf{pds}(\Gamma)$ *is defined by:*

$$\mathrm{pop}_j(\varepsilon) \textit{ is undefined}, \ \mathrm{pop}_1(a[\omega_1]\omega) = \omega, \ \mathrm{pop}_{j+1}(a[\omega_1]\omega) = a[\mathrm{pop}_j(\omega_1)]\omega.$$

Definition 5 (The push **operation at level** j**).** *For* $\alpha = bc \in \Gamma^+$, $\mathrm{push}_{j,\alpha} : it\text{-}\mathsf{pds}(\Gamma) \to it\text{-}\mathsf{pds}(\Gamma)$.

$$\mathrm{push}_{1,\alpha}(\varepsilon) = \alpha, \ \mathrm{push}_{j+1,\alpha}(\varepsilon) \textit{ is undefined for } j \geq 1$$
$$\mathrm{push}_{1,\alpha}(a[\omega_1]\omega) = b[\omega_1]c[\omega_1]\omega, \ \mathrm{push}_{j+1,\alpha}(a[\omega_1]\omega) = a[\mathrm{push}_{j,\alpha}(\omega_1)]\omega.$$

Example 6. *Given* ω_{ex} *the* 3*-pds defined in example 2:*
$\mathrm{top}(\omega_{ex}) = a_3a_2a_1$,
$\mathrm{pop}_1(\omega_{ex}) = a_3a_3[a_2[a_1b_1]]$,
$\mathrm{pop}_2(\omega_{ex}) = b_3[a_2[a_1]]a_3a_3[a_2[a_1b_1]]$, $\mathrm{pop}_3(\omega_{ex}) = b_3[b_2[a_1]a_2[a_1]]a_3a_3[a_2[a_1b_1]]$,
$\mathrm{push}_{1,a_3a_3}(\omega_{ex}) = a_3[b_2[b_1a_1]a_2[a_1]]a_3a_3[a_2[a_1b_1]]a_3[b_2[b_1a_1]a_2[a_1]]a_3a_3[a_2[a_1b_1]]$,
$\mathrm{push}_{2,a_2c_2}(\omega_{ex}) = a_3[a_2[b_1a_1]c_2[b_1a_1]a_2[a_1]]a_3a_3[a_2[a_1b_1]]$,
$\mathrm{push}_{3,a_1b_1}(\omega_{ex}) = b_3[b_2[a_1b_1a_1]a_2[a_1]]a_3[a_2[a_1b_1]]$.

A last operation will be used to describe iterated-pushdowns:

Definition 7 (Projection). *The map* $\mathrm{p}_{k,i} : k\text{-}\mathsf{pds}(\Gamma) \to i\text{-}\mathsf{pds}(\Gamma)$*, with* $1 \leq i \leq k$ *is defined by*

$$\mathrm{p}_{k,i}(\varepsilon) = \varepsilon \ \ \mathrm{p}_{k,k}(\omega) = \omega \textit{ and } \mathrm{p}_{k,i}(a[\omega_1]\omega) = \mathrm{p}_{k-1,i}(\omega_1) \textit{ if } i < k.$$

The double subscript notation will be used to handle inverse functions, the rest of the time, we will note p_i instead of $\mathrm{p}_{k,i}$.

Example 8. *Let* ω_{ex} *be the* 3*-pds given in example 2:*
$\mathrm{p}_2(\omega_{ex}) = b_2[b_1a_1]a_2[a_1]$, $\mathrm{p}_1(\omega_{ex}) = b_1a_1$.

1.1.2 Iterated pushdown automata and extensions

We extend the definition of Iterated pushdown automata used in [DG86] by allowing membership tests on the store. For $k \geq 0$, the set of level k instructions over Γ is $\mathcal{I}_k(\Gamma) = \{\mathrm{pop}_i\}_{i \in [1,k]} \cup \{push_{i,ab}\}_{a,b \in \Gamma, i \in [1,k]}$.

Definition 9 (Iterated pushdown automata). *Let* $k \geq 0$*, a* k*-pda over a terminal alphabet* Σ *is a structure* $\mathcal{A} = (Q, \Sigma, \Gamma, \vec{C}, \delta, q_0, Z)$ *where* Q *is a finite set of states,* Γ *is a pushdown alphabet with* $Z \in \Gamma$ *as initial symbol,* $\vec{C} = (C_1, \ldots, C_m)$ *is a vector of* controllers $C_i \subseteq k\text{-}\mathsf{pds}(\Gamma)$*,* $q_0 \in Q$ *is the initial state, and* $\Delta \subseteq Q \times \Sigma \times \Gamma^{(k)} - \{\varepsilon\} \times \{0,1\}^m \times \mathcal{I}_k(\Gamma) \times Q$ *is a finite set of transitions.*
The family of all k*-pdas controlled by* $\vec{C}$ *is* $k\text{-}\mathsf{PDA}(\Gamma)^{\vec{C}}$*. The set of* configurations *of* $\mathcal{A}$ *is* $Con_{\mathcal{A}} = Q \times k\text{-}\mathsf{pds}(\Gamma)$*. The single step relation* $\to_{\mathcal{A}} \subseteq Con_{\mathcal{A}} \times Con_{\mathcal{A}}$ *of* $\mathcal{A}$ *is defined by*

$$(p, \alpha w, \omega) \to_{\mathcal{A}} (q, w, \omega') \textit{ iff } (p, \alpha, \mathrm{top}(\omega), \chi_{\vec{C}}(\omega), instr, q) \in \Delta, \textit{ and } \omega' = instr(\omega),$$

where $\chi_{\vec{C}}(\omega)$ *is the boolean vector* $(o_1, \ldots, o_m)$ *fulfilling [*$o_i = 1$ *iff* $\omega \in C_i$*],* $\forall i \in [1, n]$*. We denote by* $\xrightarrow{*}_{\mathcal{A}}$ *the reflexive and transitive closure of* $\to_{\mathcal{A}}$*. The* language recognized by $\mathcal{A}$ *is* $\mathrm{L}(\mathcal{A}) = \{w \in \Sigma^* \mid \exists q \in F,\ (q_0, w, Z) \xrightarrow{*}_{\mathcal{A}} (q, \varepsilon, \varepsilon)\}$.

Example 10. *Let* $\Gamma = \{a, Z\}$*, the following automaton* $\mathcal{A} \in 2\text{-}\mathsf{PDA}(\Gamma)$ *fulfills :* $\mathrm{L}(\mathcal{A}) = \{\alpha^n\beta^n\gamma^n,\ n \geq 1\}$.

$\mathcal{A} = (\{q_0, q_1, q_2\}, \{\alpha, \beta, \gamma\}, \Gamma, \vec{\emptyset}, \Delta, q_0, Z)$ *with:*
$\Delta(q_0, \alpha, Z) = (\mathrm{push}_{2,aZ}, q_0)$, $\Delta(q_0, \alpha, Za) = (\mathrm{push}_{2,aa}, q_0)$, $\Delta(q_0, \varepsilon, Za) = (\mathrm{push}_{1,ZZ}, q_1)$,

$\Delta(q_1, \beta, Za) = (\mathrm{pop}_2, q_1)$, $\Delta(q_1, \varepsilon, ZZ) = (\mathrm{pop}_1, q_2)$,

$\Delta(q_2, \gamma, Za) = (\mathrm{pop}_2, q_2)$, $\Delta(q_2, \varepsilon, ZZ) = (\mathrm{pop}_1, q_2)$.

Here is the computation of the word $\alpha^2\beta^2\gamma^2$:
$(q_0, \alpha^2\beta^2\gamma^2, Z[\varepsilon]) \to (q_0, \alpha\beta^2\gamma^2, Z[aZ]) \to (q_0, \beta^2\gamma^2, Z[aaZ]) \to (q_1, \beta^2\gamma^2, Z[aaZ]Z[aaZ])$
$\to (q_1, \beta\gamma^2, Z[aZ]Z[aaZ]) \to (q_1, \gamma^2, Z[Z]Z[aaZ]) \to (q_2, \gamma^2, Z[aaZ]) \to (q_2, \gamma, Z[aZ]) \to$
$(q_2, \varepsilon, Z[Z]) \to (q_2, \varepsilon, \varepsilon)$.

Example 11. *Let $\Gamma = \{a, b, Z\}$, and $C = \{b^n a^n Z \in \text{1-pds}(\Gamma) \mid n \geq 1\}$. The following automaton $\mathcal{A} \in \text{1-PDA}^C(\Gamma)$ fulfills : $\mathrm{L}(\mathcal{A}) = \{\alpha^n\beta^n\gamma^n,\ n \geq 1\}$.*

$\mathcal{A} = (\{q_0, q_1\}, \{\alpha, \beta, \gamma\}, \Gamma, C, \Delta, q_0, Z)$ *with:*
$\Delta(q_0, \alpha, x, 0) = (\text{push}_{ax}, q_0)$, $x \in \{a, Z\}$,
$\Delta(q_0, \beta, x, 0) = (\text{push}_{bx}, q_0)$, $x \in \{a, b\}$,
$\Delta(q_0, \varepsilon, Z, 0) = \Delta(q_0, \varepsilon, Z, 1) = (\text{pop}_1, q_0)$,
$\Delta(q_0, \gamma, b, 1) = \Delta(q_1, \gamma, b, 1) = \Delta(q_0, \varepsilon, Z, 0) = \Delta(q_0, \varepsilon, a, 1) = (\text{pop}_1, q_1)$.

1.2 Logic

1.2.1 Monadic Second Order Logic

Let Sig be a signature and $Var = \{x, y, z, \ldots, X, Y, Z \ldots\}$ be a set of variables, where $x, y, \ldots$ denote first order variables and $X, Y, \ldots$ second order variables. The set MSO(Sig) of MSO-formulas over Sig is the smallest set such that:

- $x \in X$ and $Y \subseteq X$ are MSO-formulas for every $x, Y, X \in Var$
- $r(x_1, \ldots x_\rho)$ is an MSO-formula for every $r \in Sig$, of arity ρ and every first order variables $x_1, \ldots x_\rho \in Var$
- if Φ, Ψ are MSO-formulas then $\neg\Phi$, $\Phi \vee \Psi$, $\exists x.\Phi$ and $\exists X.\Phi$ are MSO-formulas.

Let $\mathcal{S} = \langle D_\mathcal{S}, r_1, \ldots, r_n\rangle$ be a structure over the signature Sig, a valuation of Var over $D_\mathcal{S}$ is a function $val : Var \to D_\mathcal{S} \cup \mathcal{P}(D_\mathcal{S})$ such that for every $x, X \in Var$, $val(x) \in D_\mathcal{S}$ and $val(X) \subseteq D_\mathcal{S}$. The satisfiability of an MSO-formula in the structure $\mathcal{S}$ with valuation val is then defined by induction on the structure of the formula, in the usual way.
An MSO-formula $\Phi(\bar{x}, \bar{X})$ (where $\bar{x} = (x_1, \ldots, x_\rho)$ and $\bar{X} = (X_1, \ldots, X_\tau)$ denote free first and second order variables of Φ) over Sig is said to be **satisfiable in** $\mathcal{S}$ if there exists a valuation val such that $\mathcal{S}, val \models \Phi(\bar{x}, \bar{X})$.
We will often abbreviate $\mathcal{S}, [\bar{x} \mapsto \bar{a}, \bar{X} \mapsto \bar{A}] \models \Phi(\bar{x}, \bar{X})$ by $\mathcal{S} \models \Phi(\bar{a}, \bar{A})$.

Definition 12. *A structure $\mathcal{S}$ admits a decidable MSO-theory if for every MSO-sentence Φ (i.e. MSO-formula without free variables) one can effectively decide whether $\mathcal{S} \models \Phi$.*

A subset D of $D_\mathcal{S}$ is said to be **MSO-definable** in $\mathcal{S}$ iff there exists $\phi(X)$ in MSO(Sig) such that:

$$\mathcal{S} \models \Phi(D) \text{ and } \forall S \subseteq D_\mathcal{S}, \text{ if } \mathcal{S} \models \Phi(S) \text{ then } S = D\text{s}.$$

$Sig = \{r_1, \ldots, r_n\}$ (resp. $Sig' = \{r'_1, \ldots, r'_m\}$) be some relational signature and $\mathcal{S}$ (resp. $\mathcal{S}'$) be some structure over the signature Sig (resp. Sig').

Definition 13 (Interpretations). *An* MSO-interpretation *of the structure $\mathcal{S}$ into the structure $\mathcal{S}'$ is an injective map $f : D_\mathcal{S} \to D_{\mathcal{S}'}$ such that,*

1. *$f(D_\mathcal{S})$ is* MSO*-definable in $\mathcal{S}'$*
2. *$\forall i \in [1, n]$, there exists $\Phi'_i(\bar{x}) \in MSO(Sig')$, (where $\bar{x} = x_1, \ldots, x_{\rho_i}$) fulfilling that, for every valuation val of Var in $D_\mathcal{S}$*

$$(\mathcal{S}, val) \models r_i(\bar{x}) \Leftrightarrow (\mathcal{S}', f \circ val) \models \Phi'_i(\bar{x}).$$

Theorem 14 ([Han77]). *Suppose there exists a computable* MSO**-interpretation** *of the structure $\mathcal{S}$ into the structure $\mathcal{S}'$. If $\mathcal{S}'$ has a decidable* MSO*-theory, then $\mathcal{S}$ has a decidable* MSO*-theory too.*

1.3 Logic over iterated-pushdowns

Computations of an automaton in k-$\mathsf{PDA}(\Gamma)$ are naturally expressed in the following structure $\mathsf{PDS}_k(\Gamma)$.

Definition 15. *Let* Γ *be a finite alphabet and* k *a natural number. We define the structure* $\mathsf{PDS}_k(\Gamma)^{\vec{C}}$ *by:*

$$\mathsf{PDS}_k(\Gamma) = \langle k\text{-}\mathsf{pds}(\Gamma), (\text{TOP}_u)_{u\in\Gamma^{(k)}}, (\text{POP}_i)_{i\in[1,k]}, (\text{PUSH}_{i,ab})_{i\in[1,k],a,b\in\Gamma}\rangle.$$

Relations POP_i, $\text{PUSH}_{i,u}$ *and* TOP_u *are graphs of the corresponding instructions on pushdowns.*

Computations of an automaton in $\mathsf{PDS}_k(\Gamma)^{\vec{C}}$ are expressed in an extended structure:

Definition 16. *Let* Γ *be a finite alphabet,* $k \geq 1$ *and* $\vec{C} = (C_1, \dots, C_n)$*,* $C_i \in k\text{-}\mathsf{pds}(\Gamma)$*, the structure* $\mathsf{PDS}_k(\Gamma)^{\vec{C}}$ *is obtained from* $\mathsf{PDS}_k(\Gamma)$ *by adding monadic relations* $C_1, \dots, C_n$*.*

Theorem 17. *[Fra05b], [Fra05a](Thm 6.2.2) If* $\vec{R}$ *is a vector of subsets of* Γ^**, and the MSO-theory of* $\langle\Gamma^*, (\text{SUCC}_a)_{a\in\Gamma}, \vec{R}\rangle$ *is decidable, the MSO-theory of* $\mathsf{PDS}_k(\Gamma)^{\text{p}_{k,1}{}^{-1}(\vec{R})}$ *is decidable.*

Corollary 18. *If* $\vec{R}$ *is a vector of subsets of* Γ^**, and the MSO-theory of* $\langle\Gamma^*, (\text{SUCC}_a)_{a\in\Gamma}, \vec{R}\rangle$ *is decidable, the computation graph of an automaton in* k-$\mathsf{PDA}(\Gamma)^{\text{p}_{k,1}{}^{-1}(\vec{R})}$ *has a decidable* MSO*-theory.*

1.4 Sequences

A *sequence* of natural numbers is any map $u : \mathbb{N} \to \mathbb{N}$. Such a sequence u can be also viewed as a formal power series

$$u(X) = \sum_{n=0}^{\infty} u_n X^n.$$

The following operators on series are classical:
E: the *shift* operator

$$(\mathsf{E}u)(n) = u(n+1); (\mathsf{E}u)(X) = \frac{u(X) - u(0)}{X}$$

Δ: the difference operator

$$(\Delta u)(n) = u(n+1) - u(n); (\Delta u)(X) = \frac{u(X)(1-X) - u(0)}{X}$$

Σ: the summation operator

$$(\Sigma u)(n) = \sum_{j=0}^{n} u(j); (\Sigma u)(X) = \frac{u(X)}{1-X}$$

$+$: the sum operator

$$(u+v)(n) = u(n) + v(n); (u+v)(X) = u(X) + v(X)$$

$\cdot$: the external product, for every $r \in \mathbb{Q}$

$$(r \cdot u)(n) = r \cdot u(n)$$

$\odot$: the Hadamard product, (also called the "ordinary" product)

$$(u \odot v)(n) = u(n) \cdot v(n)$$

$\times$: the convolution product

$$(u \times v)(n) = \sum_{k=0}^{n} u(k) \cdot v(n-k); (u \times v)(X) = u(X) \cdot v(X)$$

$\circ$: the sequence composition

$$(u \circ v)(n) = u(v(n))$$

$\bullet$: the series composition : if $v(0) = 0$,

$$(u \bullet v)(X) = \sum_{n=0}^{\infty} u(n) \cdot v(X)^n.$$

2 Sequences defined by automata

We define here a class of *integer sequences* by means of k-pushdown automata. Specially, we use a slightly restrictive class of k-pdas, the counter k-pdas. These are an extension of the classical *counter pda* which recognize some words with a memory consisting of natural integers only. We show that the class of sequences thus defined is closed under many natural operations.

Definition 19 (Counter k-pushdown store). *Let Γ be an alphabet with a distinguished symbol $c \in \Gamma$. The set of* k-counter pushdown stores *over Γ, with counter c, is denoted $k\text{-}\mathsf{cpds}(\Gamma)$ and defined by:*

$$1\text{-}\mathsf{cpds}(\Gamma) = (c[\varepsilon])^* \quad k+1\text{-}\mathsf{cpds}(\Gamma) = (\Gamma \cdot [k\text{-}\mathsf{cpds}(\Gamma)])^*.$$

In other words, no other symbols than c can occur at level k.

Definition 20 (Counter controlled pushdown automata). *Let $k \geq 1$ and $\vec{N} = (N_1, \ldots, N_n)$ where N_i is a subset of $\mathbb{N}$. A counter k-pda with counter controlled by $\vec{N}$, with counter c, is a k-pda $\mathcal{A} = (Q, \Sigma, \Gamma, \vec{C}, \Delta, q_0, Z)$ where $\Gamma \supseteq \{c\}$, $\vec{C} = (C_1, \ldots, C_n)$ with $C_i = \{\omega \in k\text{-}\mathsf{cpds}(\Gamma) \mid |\mathrm{p}_1(\omega)| \in N_i\}$ and such that for every $q, q' \in Q, \omega, \omega' \in k\text{-}\mathsf{pds}(\Gamma), u, u' \in X^*$, if $\omega \in k\text{-}\mathsf{cpds}(\Gamma)$ and $(q, u, \omega) \rightarrow_{\mathcal{A}} (q', u', \omega')$ then $\omega' \in k\text{-}\mathsf{cpds}(\Gamma)$.*

Then, the controller $\vec{C}$ tests whether the counter of the current memory belongs to the components of $\vec{N}$. In the rest of the paper we abbreviate "deterministic counter k-pushdown automaton" by k-dcpda.

Definition 21 ($(k, \vec{N})$-computable sequences). *Let $\vec{N}$ a vector of subsets of $\mathbb{N}$. A sequence of natural integers s is called a $(k, \vec{N})$-*computable sequence *iff there exists $\mathcal{A} \in k\text{-}\mathsf{ACD}(\Gamma)^{\vec{N}}$, over a pushdown-alphabet Γ containing at least k different symbols $a_1, a_2, \ldots, a_{k-1}, c$, with counter c, such that, for all $n \geq 0$:*

$$(q_0, \alpha^{s(n)}, a_1[a_2 \ldots [a_{k-1}[c^n]] \ldots]) \xrightarrow{*}_{\mathcal{A}} (q_0, \varepsilon, \varepsilon).$$

One denotes by $\mathbb{S}_k^{\vec{N}}$ the set of all $(k, \vec{N})$-computable sequences of natural integers (or $\mathbb{S}_k$ if $\vec{N} = \vec{\emptyset}$).

This computation scheme allows to define many recurrences. Let us expose the principle with a simple example

Example 22 (linear recurrence). *Let s be the sequence defined by*

$$s(0) = 2; \;\; \forall n \geq 0, \; s(n+1) = 2s(n) + 1.$$

Suppose there exists $\mathcal{A} \in 2\text{-}\mathsf{ACD}$ such that:

1. $(q_0, \alpha^{s(0)}, a_2[\varepsilon]) \xrightarrow{*}_{\mathcal{A}} (q_0, \varepsilon, \varepsilon)$,
2. $\forall n \geq 0$, $\forall \omega \in 2\text{-}\mathsf{pds}$, $(q_0, \varepsilon, a_2[{a_1}^{n+1}]\omega) \xrightarrow{*}_{\mathcal{A}} (q_0, \varepsilon, b_2[{a_1}^n]a_2[{a_1}^n]a_2[{a_1}^n]\omega)$,
3. $\forall n \geq 0$, $\forall \omega \in 2\text{-}\mathsf{pds}$, $(q_0, \alpha, b_2[{a_1}^n]\omega) \xrightarrow{*}_{\mathcal{A}} (q_0, \varepsilon, \omega)$.

Let us check by induction over $n \geq 0$ such an automaton fulfills the following property $\mathbf{P}(n)$: $\forall \omega \in 2\text{-}\mathsf{pds}$,

$$(q_0, \alpha^{s(n)}, a_2[{a_1}^n]\omega) \xrightarrow{*}_{\mathcal{A}} (q_0, \varepsilon, \omega).$$

Hypothesis (1) proves $\mathbf{P}(0)$*. Suppose* $\mathbf{P}(n)$ *for* $n \geq 0$*. For every* $\omega \in 2\text{-}\mathsf{pds}$*, we obtain by applying hypothesis (2), hypothesis (3), then two times* $\mathbf{P}(n)$*:*

$$\begin{array}{lll}(q_0, \alpha^{s(n+1)}, a_2[{a_1}^{n+1}]\omega) & \xrightarrow{*}_{\mathcal{A}} & (q_0, \alpha^{s(n+1)}, b_2[{a_1}^n]a_2[{a_1}^n]a_2[{a_1}^n]\omega) \\ & \xrightarrow{*}_{\mathcal{A}} & (q_0, \alpha^{s(n+1)-1}, a_2[{a_1}^n]a_2[{a_1}^n]\omega) \\ & \xrightarrow{*}_{\mathcal{A}} & (q_0, \alpha^{s(n+1)-s(n)-1}, a_2[{a_1}^n]\omega) \\ & \xrightarrow{*}_{\mathcal{A}} & (q_0, \varepsilon, \omega).\end{array}$$

Then, $\mathbf{P}(n)$ *is true for every* $n \geq 0$*, and in the particular case where* $\omega = \varepsilon$*,* $\mathcal{A}$ *computes the sequence* s*.*

Let us prove there exists an automaton in $2\text{-}\mathsf{ACD}$ *fulfilling hypothesis (1), (2) and (3). Let* $\mathcal{A} = (\{q_0, q_1\}, \{\alpha\}, \Gamma, \Delta, q_0, Z)$ *where* $\Gamma = \{a_1, a_2, b_2, Z\}$ *and:*

(a) $\Delta(q_0, \varepsilon, a_2) = (\text{push}_{1,b_2b_2}, q_0)$,

(b) $\Delta(q_0, \varepsilon, a_2a_1) = (\text{pop}_1\,\text{push}_{1,a_2a_2}, q_1)$ *and* $\Delta(q_0, \varepsilon, a_2) = \Delta(q_0, \varepsilon, a_2a_1) = (\text{push}_{1,b_2a_2}, q_0)$,

(c) $\delta(q_0, \alpha, b_2) = \delta(q_0, \alpha, b_2a_1) = (\text{pop}_2, q_0)$.

This automaton is deterministic, transitions (a) and (c) allow the computation given hypothesis (1), transitions (b) makes true hypothesis (2), and transition (c) allows the calculus (3).

Proposition 23. *For every* $s \in \mathbb{S}_k^{\vec{N}}$*, one can construct* $\mathcal{A}_1 \in k\text{-}\mathsf{AC}^{\vec{N}}$*, such that* $\mathrm{L}(\mathcal{A}_1) = \{\alpha^{s(n)} \mid n \geq 0\}$.

2.1 Some computable sequences

Definition 24 ($\mathbb{N}$-rational sequences). *A sequence* $(u_n)_{n\geq 0}$ *is* $\mathbb{N}$*-rational iff there is a matrix* M *in* $\mathbb{N}^{d\times d}$ *and two vectors* L *in* $\mathbb{B}^{1\times d}$ *and* C *in* $\mathbb{B}^{d\times 1}$ *such that* $u_n = L \cdot M^n \cdot C$*.*

Proposition 25 ([FS03]). *If* $(u_n)_{n\geq 0}$ *is* $\mathbb{N}$*-rational, then* $(u_n)_{n\geq 0} \in \mathbb{S}_2$*.*

Proposition 26 ([FS03]). *Let* $P_i(X_1, \ldots, X_p)$*,* $(1 \leq i \leq p)$ *be polynomials with coefficients in* $\mathbb{N}$*,* $c_1, \ldots, c_i, \ldots c_p \in \mathbb{N}$ *and ,* u_i $(1 \leq i \leq p)$ *be the sequence defined by* $u_i(n+1) = P_i(u_1(n), \ldots, u_p(n))$*, and* $u_i(0) = c_i$*. Then* $u_1 \in \mathbb{S}_3$*.*

Proposition 27. *Let* s *be a strictly increasing sequence such that* $s(0) = 0$*, then* $s^{-1} \in \mathbb{S}_2^{s(\mathbb{N})}$*.*

Proof: $\mathcal{A} = (\{q_0\}, \{\alpha\}, (\{a_1\}, \{a_2\}), s(\mathbb{N}), \Delta, q_0)$ with
$\Delta(q_0, \varepsilon, a_2, o) = (q_0, \text{pop}_2)$ for $o \in \{0, 1\}$,
$\Delta(q_0, \varepsilon, a_2a_1, 0) = \Delta(q_0, \alpha, a_2a_1, 1) = (\text{pop}_1, q_0)$.

Starting from a configuration $(q_0, \sigma, a_2[{a_1}^n])$, $\mathcal{A}$ pops iteratively the level 1, by reading to each iteration a terminal letter α iff the counter belongs to $s(\mathbb{N})$. Finally, when the level 1 remains empty, the length of the terminal word read is the number of elements of $[1, n] \cap s(\mathbb{N})$, i.e., $s^{-1}(n)$. □

Theorem 28.

0- For every $f \in \mathbb{S}_{k+1}^{\vec{N}}$*,* $k \geq 1$*, and every integer* $c \in \mathbb{N}$*, sequences* Ef *and* $f + \frac{c}{1-X}$*, belong to* $\mathbb{S}_{k+1}^{\vec{N}}$*; if* $\forall n \in \mathbb{N}, f(n) \geq c$ *then* $f - \frac{c}{1-X}$ *belongs to* $\mathbb{S}_{k+1}^{\vec{N}}$*; the sequence* $0 \mapsto c, n+1 \mapsto f(n)$ *belongs to* $\mathbb{S}_{k+1}^{\vec{N}}$*.*

1- For every $f, g \in \mathbb{S}_{k+1}^{\vec{N}}$*, with* $k \geq 1$*, the sequence* $f + g$ *belongs to* $\mathbb{S}_{k+1}^{\vec{N}}$*.*

2- For every $f, g \in \mathbb{S}_{k+1}^{\vec{N}}$*, with* $k \geq 2$*, the sequence* $f \odot g$*, belongs to* $\mathbb{S}_{k+1}^{\vec{N}}$ *and for every* $f' \in \mathbb{S}_{k+2}^{\vec{N}}$*,* f'^g *belongs to* $\mathbb{S}_{k+2}^{\vec{N}}$*.*

3- For $f \in {\mathbb{S}_{k+1}}^{\vec{N}}$*,* $g \in \mathbb{S}_k$*,* $k \geq 2$*, sequences* $f \times g$ *and* $f \bullet g$ *belong to* $\mathbb{S}_{k+1}^{\vec{N}}$*.*

4- For every $g \in \mathbb{S}_k$*, with* $k \geq 2$*, the sequence* f *defined by:* $f(n+1) = \sum_{m=0}^{n} f(m) \cdot g(n-m)$ *and* $f(0) = 1$ *(the convolution inverse of* $1 - X \times f$*) belongs to* $\mathbb{S}_{k+1}$*.*

5- For every $f \in \mathbb{S}_k$*,* $g \in \mathbb{S}_\ell^{\vec{N}}$*, for* $k, l \geq 2$*, the sequence* $f \circ g$ *belongs to* $\mathbb{S}_{k+\ell-1}^{\vec{N}}$*.*

6- For every $k \geq 2$ *and for every system of recurrent equations expressed by polynomials in* $\mathbb{S}_{k+1}^{\vec{N}}[X_1, \ldots, X_p]$*, with initial conditions in* $\mathbb{N}$*, every solution belongs to* $\mathbb{S}_{k+1}^{\vec{N}}$*.*

7- For every $k \geq 2$ and for every every system of recurrent equations expressed by polynomials with undetermined $X_1, \ldots, X_p$, coefficients in $\mathbb{S}^{\vec{N}}_{k+2}$, exponents in $\mathbb{S}^{\vec{N}}_{k+1}$ and initial conditions in $\mathbb{N}$, every solution belongs to $\mathbb{S}^{\vec{N}}_{k+2}$.

An analogous result is proved in [FS03] for sequences in $\mathbb{S}_k$. Except some technical parts, the proof of Theorem 28 is essentially the same.

3 Application to the sequential calculus

We use here decidability results on k-pdas in order to demonstrate the decidability of the monadic theory of structures $\langle \mathbb{N}, +1, P\rangle$, for a large class of predicates P (Theorem 29 and Theorem 36) containing for example $(n\lfloor\sqrt{n}\rfloor)_{n\in\mathbb{N}}$ or $(n\lfloor\log n\rfloor)_{n\in\mathbb{N}}$. These results can be generalised to the case of structures with several nested predicates (Theorem 32), as for example

$$\langle \mathbb{N}, +1, \{n^{k_1}\}_{n\geq 0}, \{n^{k_1 k_2}\}_{n\geq 0}, \ldots, \{n^{k_1\cdots k_m}\}_{n\geq 0}\rangle, \text{ for } k_1, \ldots, k_m \geq 0.$$

3.1 Extensions of $\langle \mathbb{N}, +1\rangle$

It is proved in [FS03] that for every sequence s calculated by a k-dcpda $\mathcal{A}$ (in the sense of Definition 21), the structure $\langle \mathbb{N}, +1, \Sigma s(\mathbb{N})\rangle$ is interpretable inside the computation graph of $\mathcal{A}$. According to Corollary 18, this graph has a decidable MSO-theory.

Theorem 29 ([FS03]). *For every $s \in \mathbb{S}_k$, $k \geq 1$, the* MSO*-theory of $\langle \mathbb{N}, +1, \Sigma s(\mathbb{N})\rangle$ is decidable.*

In the same way, we can prove that for every sequence s calculated by a $\mathcal{A} \in k\text{-}\mathsf{ACD}^{\vec{N}}$ (in the sense of Definition 21), the structure $\langle \mathbb{N}, +1, \Sigma s(\mathbb{N})\rangle$ is interpretable inside the computation graph of $\mathcal{A}$. Using Corollary 18, we obtain then:

Theorem 30. *If $s \in \mathbb{S}^{\vec{N}}_k$, with $\vec{N} = (N_1, \ldots, N_m)$ such that $\langle \mathbb{N}, +1, N_1, \ldots, N_m\rangle$ has a decidable* MSO*-theory, then $\langle \mathbb{N}, +1, \Sigma s(\mathbb{N})\rangle$ has a decidable* MSO*-theory.*

Corollary 31. *Structures $\langle \mathbb{N}, +1, (n\lfloor\sqrt{n}\rfloor)_{n\in\mathbb{N}}\rangle$, and $\langle \mathbb{N}, +1, (n\lfloor\log n\rfloor)_{n\in\mathbb{N}}\rangle$ have a decidable* MSO*-theory.*

Proof: Let us describe the proof for $n\lfloor\sqrt{n}\rfloor$. Consider the sequence s defined for $n \geq 0$ by

$$\begin{cases} \lfloor\sqrt{n}\rfloor) & \text{if } n \notin \{m^2\}_{m\geq 0} \\ \lfloor\sqrt{n}\rfloor + n & \text{if } n \in \{m^2\}_{m\geq 0}. \end{cases}$$

Then $\Sigma s = (n\lfloor\sqrt{n}\rfloor)_{n\geq 0}$. The sequence s belongs to $\mathbb{S}_2^{\{m^2\}_{m\geq 0}}$. Indeed, by using Lemma 27, it is possible to construct two automata $\mathcal{A}_1$ and $\mathcal{A}_2 \in 2\text{-}\mathsf{ACD}^{\{m^2\}_{m\geq 0}}$ such that $\mathcal{A}_1$ computes $\lfloor\sqrt{n}\rfloor$ from $(q_0, a_2[c^n])$ and $\mathcal{A}_2$ computes $\lfloor\sqrt{n} + n\rfloor$ from $(q_0, b_2[c^n])$. Using the controller $\{m^2\}_{m\geq 0}$, it is easy to compose these automata to construct $\mathcal{B} \in 2\text{-}\mathsf{ACD}^{\{m^2\}_{m\geq 0}}$ calculating $s(n)$.

Then $(n\lfloor\sqrt{n}\rfloor)_{n\geq 0}$ belongs to $\Sigma\mathbb{S}_2^{\{m^2\}_{m\geq 0}}$ and $\langle \mathbb{N}, +1, (m^2)_{m\in\mathbb{N}}\rangle$ has a decidable MSO-theory (see [ER66]). By applying Theorem 30, the MSO-theory of $\langle \mathbb{N}, +1, (n\lfloor\sqrt{n}\rfloor)_{n\geq 0}\rangle$ is decidable.

For the sequence $(n\lfloor\log n\rfloor)_{n\geq 0}$, we proceed in the same way, by using the fact that $\langle \mathbb{N}, +1, (2^n)_{n\in\mathbb{N}}\rangle$ has a decidable MSO-theory (see [ER66]). □

Theorem 32. *If $s \in \mathbb{S}^{\vec{N}}_k$, with $\vec{N} = (N_1, \ldots, N_m)$ such that $\langle \mathbb{N}, +1, N_1, \ldots, N_m\rangle$ has a decidable* MSO*-theory, then $\langle \mathbb{N}, +1, \Sigma s(\mathbb{N}), \Sigma s(N_1), \ldots, \Sigma s(N_m)\rangle$ has a decidable* MSO*-theory.*

Proof: It is possible to construct a $k\text{-}\mathsf{ACD}^{\vec{N}}$ recognizing the language $L \in (\{\alpha\} \cup \{\beta_{\vec{\sigma}} \mid \vec{\sigma} \in \{0,1\}^m\})^*$:

$$L = \{\alpha^{s(0)} x_0 \cdots \alpha^{s(n)} x_n \mid n \geq 0, \forall i \in [1, n], x_i = \beta_{\chi_{\vec{N}}(i)}\}$$

and whose computation graph consists of an infinite path labelled by the word

$$\alpha^{s(0)}\beta_{\chi_{\vec{N}}(0)}\cdots\alpha^{s(n)}\beta_{\chi_{\vec{N}}(n)}\cdots$$

Let $P_{\vec{\sigma}} = \{n \mid \chi_{\vec{N}}(n) = \vec{\sigma}\}$. The structure $\mathcal{S} = \langle \mathbb{N}, +1, \boldsymbol{\Sigma}s(\mathbb{N}), (\boldsymbol{\Sigma}s(P_{\vec{\sigma}}))_{\vec{\sigma}\in\{0,1\}^m}\rangle$ is interpretable in this graph. From Corollary 18, since $\langle \mathbb{N}, +1, \vec{N}\rangle$ has a decidable theory, the structure $\mathcal{S}$ so has.

Finally, $\langle \mathbb{N}, +1, \boldsymbol{\Sigma}s(\mathbb{N}), \boldsymbol{\Sigma}s(N_1), \ldots, \boldsymbol{\Sigma}s(N_m)\rangle$ is clearly interpretable in $\mathcal{S}$ since for every $i \in [1, m]$,

$$\boldsymbol{\Sigma}s(N_i) = \bigcup_{\vec{\sigma}\mid\pi_i(\vec{\sigma})=1} \boldsymbol{\Sigma}s(P_{\vec{\sigma}})$$

and has then a decidable MSO-theory. □

Corollary 33. *For every integers $k_1, \ldots, k_m \geq 0$, the* MSO*-theory of the structure*

$$\langle \mathbb{N}, +1, \{n^{k_m}\}_{n\geq 0}, \{n^{k_m k_{m-1}}\}_{n\geq 0}, \ldots, \{n^{k_1\cdots k_m}\}_{n\geq 0}\rangle$$

is decidable.

Proof: Let $u_i(n) = \sum_{j=0}^{k_i}\binom{j}{k}n^j$. Clearly, $\boldsymbol{\Sigma}u_i(n) = n^{k_i}$ and from Theorem 28, $u_i \in \mathbb{S}_2$ since u_i is $\mathbb{N}$-rational. Then, for every $i \in [1, m]$, the sequence $(n^{k_i})_{n\in\mathbb{N}}$ belongs to $\boldsymbol{\Sigma}\mathbb{S}_2$.
Let us prove the corollary by induction over $m \geq 1$.
Basis: If $m = 1$, then $(n^{k_1})_{n\in\mathbb{N}} \in \boldsymbol{\Sigma}\mathbb{S}_2$, and from Theorem 29, the result holds.
Induction step: Suppose the corollary true for $m \geq 1$, and consider $\vec{N} = (N_1, \ldots, N_m)$ with $\forall i \in [1, m]$, $N_i = \{n^{k_m\cdots k_i} \mid n \geq 0\}$. The sequence u_{m+1} belongs to ${\mathbb{S}_2}^{\vec{N}}$ and Theorem 32 implies

$$\langle \mathbb{N}, +1, \boldsymbol{\Sigma}u_{m+1}(\mathbb{N}), \boldsymbol{\Sigma}s(N_1), \ldots, \boldsymbol{\Sigma}s(N_m)\rangle \text{ has a decidable MSO-theory.}$$

In addition, $\boldsymbol{\Sigma}u_{m+1}(\mathbb{N}) = \{n^{k_{m+1}}\}_{n\geq 0}$ and $\forall i \in [1, m]$,

$$\boldsymbol{\Sigma}u_{m+1}(N_i) = \{\sum_{j=0}^{j=n^{k_m\cdots k_i}} u_{m+1}(j) \mid n \geq 0\} = \{(n^{k_m\cdots k_i})^{k_{m+1}} \mid n \geq 0\}.$$

□

3.2 Differentiably, k-computable sequences

The particular form of the predicates $\boldsymbol{\Sigma}s(\mathbb{N})$ considered in Theorem 29 leads naturally to the following class of sequences.

Definition 34. *Let $k \geq 2$ and $\vec{N}$ a vector of subsets of $\mathbb{N}$. We define the class $\boldsymbol{\Sigma}\mathbb{S}_k^{\vec{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ as the set*

$$\boldsymbol{\Sigma}\mathbb{S}_k^{\vec{N}} = \{\boldsymbol{\Sigma}v \mid v \in \mathbb{S}_k^{\vec{N}}\}.$$

Remark 35. *It can be proved that classes $\boldsymbol{\Sigma}\mathbb{S}_k$ are included in the class of "residually ultimately periodic" (RUP) sequences studied by [CT02]. It is shown in [CT02] that for any RUP sequence s , the theory of $\langle \mathbb{N}, +1, s(\mathbb{N})\rangle$ is decidable. It can be proved that sequences in $\boldsymbol{\Sigma}\mathbb{S}_k^{\vec{N}}$ considered Theorem 32, like $(n\lfloor\sqrt(n)\rfloor)_{n\in\mathbb{N}}$ or $(n\lfloor\log(n)\rfloor)_{n\in\mathbb{N}}$ are not RUP.*

We show now classes $\boldsymbol{\Sigma}\mathbb{S}_k^{\vec{N}}$ are closed by many operations. The definition of the operator $\boldsymbol{\Sigma}$, as well as other classical definitions about sequences are recalled in §1.4.

Theorem 36.
0- For every $u \in \Sigma\mathbb{S}_{k+1}^{\vec{N}}$, $k \geq 1$, and every integer $c \in \mathbb{N}$, the sequences Eu, $u + \frac{c}{1-X}$ (adding c to every term), belong to $\Sigma\mathbb{S}_{k+1}^{\vec{N}}$;
if $u(n) \geq c$ then $u - \frac{c}{1-X}$ (subtracting c to every term) belongs to $\Sigma\mathbb{S}_{k+1}^{\vec{N}}$;
if $u(0) \geq c$, then the sequence $0 \mapsto c, n+1 \mapsto u(n)$ belongs to $\Sigma\mathbb{S}_{k+1}^{\vec{N}}$.
1- For every $u, v \in \Sigma\mathbb{S}_{k+1}^{\vec{N}}$, $k \geq 1$, the sequence $u + v$ belongs to $\Sigma\mathbb{S}_{k+1}^{\vec{N}}$.
2- For every $u, v \in \Sigma\mathbb{S}_{k+1}^{\vec{N}}$, $k \geq 2$, the sequence $u \odot v$ belongs to $\Sigma\mathbb{S}_{k+1}^{\vec{N}}$.
3- For every $u \in \Sigma\mathbb{S}_{k+1}^{\vec{N}}$, $v \in \Sigma\mathbb{S}_k$, $k \geq 2$, $u \times v$ belongs to $\Sigma\mathbb{S}_{k+1}^{\vec{N}}$.
4- For every $u \in \Sigma\mathbb{S}_k$, $k \geq 2$, such that $v(0) \geq 1$, the sequence u defined by: $u(0) = 1$ and $u(n+1) = \sum_{m=0}^{n} u(m) \cdot v(n-m)$ (the convolution inverse of $1 - Xv$) belongs to $\Sigma\mathbb{S}_{k+1}$.
5- For every $u \in \Sigma\mathbb{S}_k$, $v \in \Sigma\mathbb{S}_\ell^{\vec{N}}$, $k, l \geq 2$, $u \circ v$ belongs to $\Sigma\mathbb{S}_{k+\ell-1}^{\vec{N}}$.
6- For every $k \geq 2$, if $u_1(n), \ldots u_p(n)$ is the vector of solutions of a system of recurrent equations expressed by polynomials in $\Sigma\mathbb{S}_{k+1}^{\vec{N}}[X_1, \ldots, X_p]$, with initial conditions $u_i(0), u_i(1) \in \mathbb{N}$, with $u_i(0) \leq u_i(1)$, then $u_1 \in \Sigma\mathbb{S}_{k+1}^{\vec{N}}$.

Let us recall that, from Theorem 32, if $\langle \mathbb{N}, +1, N_1, \ldots, N_m \rangle$ has a decidable MSO-theory, then for every sequence $u \in \Sigma\mathbb{S}_{k+1}^{\vec{N}}$, the predicate $P = \{u(n) \mid n \in \mathbb{N}\}$ leads to a structure $\langle \mathbb{N}, +1, P \rangle$ which has a decidable Monadic Second Order theory.

References

[Bès01] Alexis Bès. A survey of arithmetical definability. *Bull. Belg. Math. Soc. Simon Stevin*, (suppl.):1–54, 2001. A tribute to Maurice Boffa.

[CT02] Olivier Carton and Wolfgang Thomas. The monadic theory of morphic infinite words and generalizations. *Inform. and Comput.*, 176(1):51–65, 2002.

[DG86] Werner Damm and Andreas Goerdt. An automata-theoretical characterization of the OI-hierarchy. *Inform. and Control*, 71(1-2):1–32, 1986.

[ER66] Calvin C. Elgot and Michael O. Rabin. Decidability and undecidability of extensions of second (first) order theory of (generalized) successor. *J. Symbolic Logic*, 31(2):169–181, 1966.

[Fra05a] S. Fratani. *Automates à piles de piles . . . de piles.* PhD thesis, Université Bordeaux 1, 2005.

[Fra05b] S. Fratani. Regular sets over tree structures. Submitted to Theoret. Comput. Sci. Available version at http://dept-info.labri.u-bordeaux.fr/˜fratani, 2005.

[FS03] S. Fratani and G. Sénizergues. Iterated pushdown automata and sequences of rational numbers. Presented to *Second St.Petersburg Days of Logic and Computability, 2003.* To appear in APAL, available version at http://dept-info.labri.u-bordeaux.fr/˜fratani, 2003.

[Gre70] Sheila A. Greibach. Full AFLs and nested iterated substitution. *Information and Control*, 16:7–35, 1970.

[Han77] *Handbook of mathematical logic*, chapter Decidable theories. M.O.Rabin, pages 595–629. North-Holland Publishing Co., Amsterdam, 1977. Edited by Jon Barwise, With the cooperation of H. J. Keisler, K. Kunen, Y. N. Moschovakis and A. S. Troelstra, Studies in Logic and the Foundations of Mathematics, Vol. 90.

[Mae99] Arnaud Maes. An automata-theoretic decidability proof for first-order theory of $\langle \mathbf{N}, <, P \rangle$ with morphic predicate P. *J. Autom. Lang. Comb.*, 4(3):229–245, 1999. Journées Montoises d'Informatique Théorique (Mons, 1998).

[Sem84] A. L. Semenov. Decidability of monadic theories. In *Mathematical foundations of computer science, 1984 (Prague, 1984)*, volume 176 of *Lecture Notes in Comput. Sci.*, pages 162–175. Springer, Berlin, 1984.

[Sie70] D. Siefkes. Decidable extensions of monadic second order successor arithmetic. In *Automatentheorie und formale Sprachen (Tagung, Math. Forschungsinst., Oberwolfach, 1969)*, pages 441–472. Bibliographisches Inst., Mannheim, 1970.